Midterm 1 / 2008.10.16 / Calculus for the Biosciences / MAT 1194.002

Note Title 10/16/2008

1. An exponentially growing yeast culture doubles in 7 days. How long would it take it to quadruple in size?

In the first 7 days you get double. In the next 7 days you get double that, which is 4 times the size.
$\therefore$ Quadrupling time is a fortnight.

Check: $b(t) = b_0 e^{kt}$ for some k, where b_0 is the initial size

Since doubling time is 7 days $2\cancel{b_0} = \cancel{b_0} e^{k7}$

$\ln 2 = k7$, so $k = \frac{\ln 2}{7}$

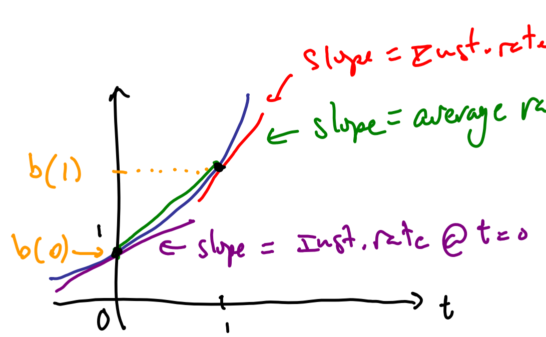
To find quadrupling time set $4\cancel{b_0} = \cancel{b_0} e^{kt}$

and solve for t: $\ln 4 = kt$

$$\therefore \quad t = \frac{\ln 4}{k} = \frac{\ln 4}{\ln 2} \cdot 7 = \frac{\ln(2^2)}{\ln 2} 7 = \frac{2 \cancel{\ln 2}}{\cancel{\ln 2}} 7 = 14$$:)

2. A population of bacteria grows exponentially according to $b(t) = e^{2t}$. Find and illustrate on a graph

(a) Population at $t = 0$ and $t = 1$.
(b) The average rate of change between $t = 0$ and $t = 1$.
(c) The instantaneous rates of change at $t = 0$ and $t = 1$.



a) $b(0) = 1$

$b(1) = e^2 \approx 7.389$

b) $\frac{b(1) - b(0)}{1 - 0} = \frac{e^2 - 1}{1} = e^2 - 1 \approx 6.389$

c) $b'(t) = 2e^{2t}$, $b'(0) = 2$, $b'(1) = 2 \cdot e^2 \approx 14.778$

3. Find the derivatives of

(a) $\cos(1 + e^{2x})$ (b) $\ln(\ln x)$

a) $[\cos(1+e^{2x})]' = -\sin(1+e^{2x}) \cdot e^{2x} \cdot 2$

b) $[\ln(\ln t)]' = \frac{1}{\ln t} \cdot \frac{1}{t}$

4. Find the second derivative of the Hill function $x^2/(1+x^2)$ and use it to describe the curvature of the Hill function's graph.

$$\text{If } h = \frac{x^2}{1+x^2}, \quad h' = \frac{(x^2)'(1+x^2) - x^2(1+x^2)'}{(1+x^2)^2}$$

$$= \frac{2x(1+x^2) - x^2 \cdot 2x}{(1+x^2)^2} = \frac{2x + \cancel{2x^3} - \cancel{2x^3}}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

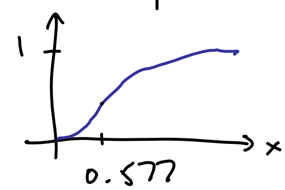
$$h'' = 2\,\frac{x'(1+x^2)^2 - x\left[(1+x^2)^2\right]'}{(1+x^2)^4}$$

← chain rule

$$= 2\,\frac{(1+x^2)^{\cancel{2}} - x\,2\cancel{(1+x^2)}\,2x}{(1+x^2)^{\cancel{4}\,3}} = 2\,\frac{1+x^2-4x^2}{(1+x^2)^3} = 2\,\frac{1-3x^2}{(1+x^2)^3}$$

For $x \geq 0$ $\quad h'' \geq 0 \iff 3x^2 \leq 1 \iff x^2 \leq \frac{1}{3} \iff x \leq \frac{1}{\sqrt{3}} \approx 0.577$

$\therefore$ h curves up until about $x = 0.577$ and then curves down



5. The amount of medication M_t in the bloodstream of a patient on an intravenous drip is governed by the discrete dynamical system $M_{t+1} = M_t - f(M_t)M_t + d$, where d is the rate of delivery through the drip and $f(M_t)$ is the fraction of the medication absorbed by the patient. If $f(M_t) = M_t/(2+M_t)$ and $d = 1$, find the biologically significant equilibrium and determine its stability.

$$M_{t+1} = M_t - \frac{M_t}{2+M_t} M_t + 1$$

To find the equilibrium set $\cancel{M} = \cancel{M} - \dfrac{M^2}{2+M} + 1$

$$\frac{M^2}{2+M} = 1 \qquad M^2 = 2+M \qquad M^2 - M - 2 = 0$$

Not significant.

By inspection $M = \cancel{-1}, 2$

(or use quadratic formula ☺)

$$\left[M - \frac{M^2}{2+M} + 1\right]' = 1 - \frac{(M^2)'(2+M) - M^2(2+M)'}{(2+M)^2}$$

$$= 1 - \frac{2M(2+M) - M^2}{(2+M)^2} = 1 - \frac{4M + 2M^2 - M^2}{(2+M)^2} = 1 - \frac{4M + M^2}{(2+M)^2}$$

Eval @ $M=2$: $1 - \dfrac{8+4}{4^2} = 1 - \dfrac{2+1}{4} = 1 - \dfrac{3}{4} = \dfrac{1}{4} < 1$

$\therefore$ $M = 2$ is a stable equilibrium