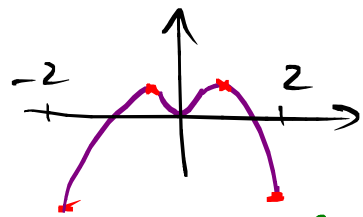
Midterm 2 / 2011.12.1 / Calculus for the Biosciences / MAT 1193.005

(checked with Maple)

Note Title 12/6/2011

1. Find all critical points of $f(x) = x^2 - x^4$ in the interval $-2 \le x \le 2$. Use f'' to determine whether they are local minima or maxima. Find the global minimum and maximum of f of the interval and state where they occur. Sketch.

$f(x) = x^2 - x^4$



$f'(x) = 2x - 4x^3 = 2x(1-2x^2)$

$f' = 0 \Rightarrow x = 0$ or $x = \pm\frac{1}{\sqrt{2}} \approx \pm 0.7 \leftarrow$ Critical points

$f''(x) = 2 - 12x^2$

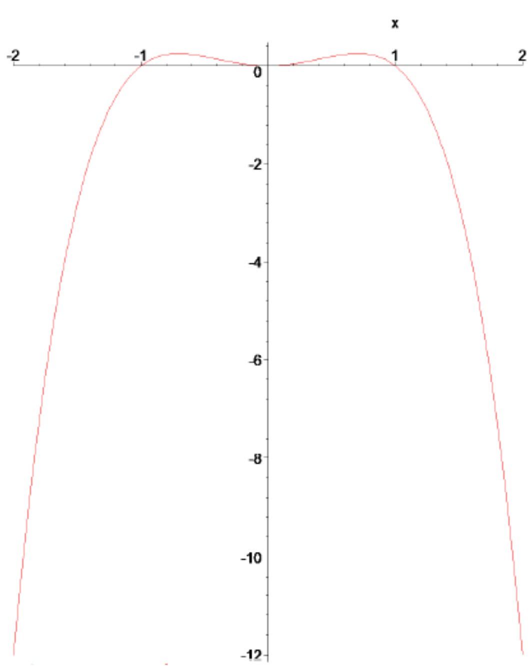
$f''(0) = 2 > 0$ $\therefore$ concave up $\therefore$ local min @ 0

$f''(\pm\frac{1}{\sqrt{2}}) = 2 - 12\cdot\frac{1}{2} = -4 < 0$ $\therefore$ concave down

$\therefore$ local maxes @ $\pm\frac{1}{\sqrt{2}}$

	x	$f(x)$	
crit.	0	0	
	$\pm\frac{1}{\sqrt{2}}$	$\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$	$\leftarrow$ global max
endpts	± 2	$4 - 16 = -12$	$\leftarrow$ global min

```
> plot(f,x=-2..2);
```




```
> Digits:=4:
> f:=x^2-x^4;
```

$f := x^2 - x^4$

```
> df:=diff(f,x);
```

$df := 2x - 4x^3$

```
> crit:=[solve(df,x)]; evalf(%);
```

$crit := \left[0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right]$

[0., 0.7070, -0.7070]

```
> ddf:=diff(df,x);
```

$ddf := 2 - 12x^2$

```
> map(v->subs(x=v,ddf),crit);
```

[2, -4, -4]

```
> convert(crit,set) union {-2,2}: convert(%,list);
  map(v->subs(x=v,f),%); evalf(%);
```

$\left[-2, 0, 2, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right]$

$\left[-12, 0, -12, \frac{1}{4}, \frac{1}{4}\right]$

[-12., 0., -12., 0.2500, 0.2500]

2. Find indefinite integrals of the following functions

$$\text{(a)}\ \frac{e^{3x}}{(1-e^{3x})^3} \qquad \text{(b)}\ \frac{\ln x}{x} \qquad \text{(c)}\ t^3\sin(2t)$$

a) $$\int \frac{e^{3x}}{(1-e^{3x})^3}\,dx = -\frac{1}{3}\int \frac{du}{u^3} = -\frac{1}{3}\int u^{-3}\,du$$

Let $u = 1 - e^{3x}$

Then $\frac{du}{dx} = -3e^{3x}$

$e^{3x}\,dx = -\frac{1}{3}\,du$

$$= -\frac{1}{3}\,\frac{u^{-3+1}}{-3+1} = \frac{1}{6}u^{-2}$$

$$= \frac{1}{6}\,\frac{1}{(1-e^{3x})^2} + C$$

b) $$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} = \frac{(\ln x)^2}{2} + C$$

Let $u = \ln x$ $\quad \frac{du}{dx} = \frac{1}{x}$ $\quad \frac{1}{x}\,dx = du$

c) $$\int t^3\sin(2t)\,dt = -\frac{t^3}{2}\cos(2t) + \frac{3}{4}t^2\sin(2t) + \frac{3}{4}t\cos(2t) - \frac{3}{8}\sin(2t) + C$$

diff. ↓		↓ int.
t^3	+	$\sin(2t)$
$3t^2$	−	$-\frac{1}{2}\cos(2t)$
$6t$	+	$-\frac{1}{4}\sin(2t)$
6	−	$\frac{1}{8}\cos(2t)$
0		$\frac{1}{16}\sin(2t)$

alternate sum of products

```
> exp(3*x)/(1-exp(3*x))^3; int(%,x);
```

$$\frac{e^{(3x)}}{(1-e^{(3x)})^3}$$

$$\frac{1}{6}\,\frac{1}{(1-e^{(3x)})^2}$$

```
> ln(x)/x; int(%,x);
```

$$\frac{\ln(x)}{x}$$

$$\frac{1}{2}\ln(x)^2$$

```
> t^3*sin(2*t); int(%,t);
```

$$t^3\sin(2t)$$

$$-\frac{1}{2}t^3\cos(2t) + \frac{3}{4}t^2\sin(2t) - \frac{3}{8}\sin(2t) + \frac{3}{4}t\cos(2t)$$

3. Determine whether the improper integral $\int_0^1 \frac{dx}{x^{\frac{1}{4}} + x^{\frac{5}{4}}}$ converges or diverges. Justify your assertion by comparison to an integral whose convergence or divergence can be determined directly.

③ $\int_0^1 \frac{dx}{x^{1/4} + x^{5/4}}$ compare to $\int_0^1 \frac{dx}{x^{1/4}}$

Limit comparison:

$$\frac{1}{x^{1/4}+x^{5/4}} \Big/ \frac{1}{x^{1/4}} = \frac{x^{1/4}}{x^{1/4}+x^{5/4}} = \frac{1}{1+\underbrace{x}_{\to 0}} \to 1 \quad \begin{matrix} \neq 0 \\ \neq \infty \end{matrix}$$

$\therefore$ comparison is good

On the other hand $\int_0^1 \frac{dx}{x^{1/4}}$ conv. (p-test $p = \frac{1}{4} < 1$)

(Or directly: $\int_0^1 x^{-1/4}\,dx = \frac{4}{3}x^{3/4}\Big|_0^1 = \frac{4}{3}$)

$\therefore$ Orig integral conv.

Alt.: direct comparison

$$\int_0^1 \frac{dx}{x^{1/4}+x^{5/4}} \le \int_0^1 \frac{dx}{x^{1/4}} = \frac{4}{3}$$

Actual value:

```
> 1/(x^(1/4)+x^(5/4)); int(%,x=0..1); evalf(%);
```

$$\frac{1}{x^{(1/4)}+x^{(5/4)}}$$

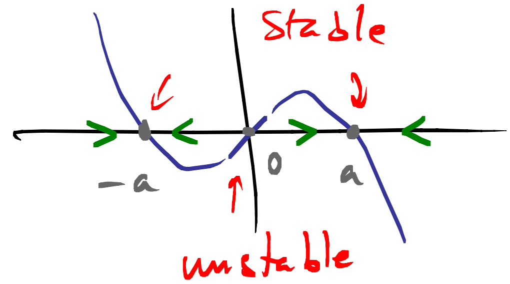
$$\frac{1}{2}\sqrt{2}\ln(2-\sqrt{2})-\frac{1}{2}\sqrt{2}\ln(2+\sqrt{2})+\frac{\sqrt{2}\,\pi}{2}$$

$$0.976$$

4. For the autonomous differential equation $dx/dt = a^2x - x^3$, where a is a positive constant, draw the phase-line diagram, find the equilibria, and determine their stability.

(4) $$\frac{dx}{dt} = a^2x - x^3 = x(a^2-x^2) = x(a-x)(a+x)$$

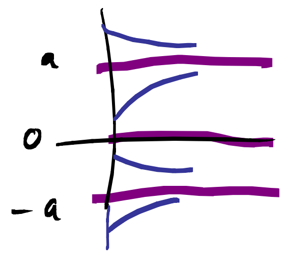
Equi: $0, \pm a$



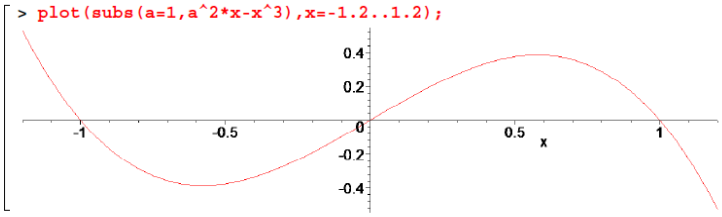
$$\frac{d}{dx}(a^2x - x^3) = a^2 - 3x^2$$

@ $0 \longrightarrow a^2 > 0 \quad \therefore$ unstable

@ $\pm a \longrightarrow a^2 - 3a^2 = -2a^2 < 0 \quad \therefore$ stable

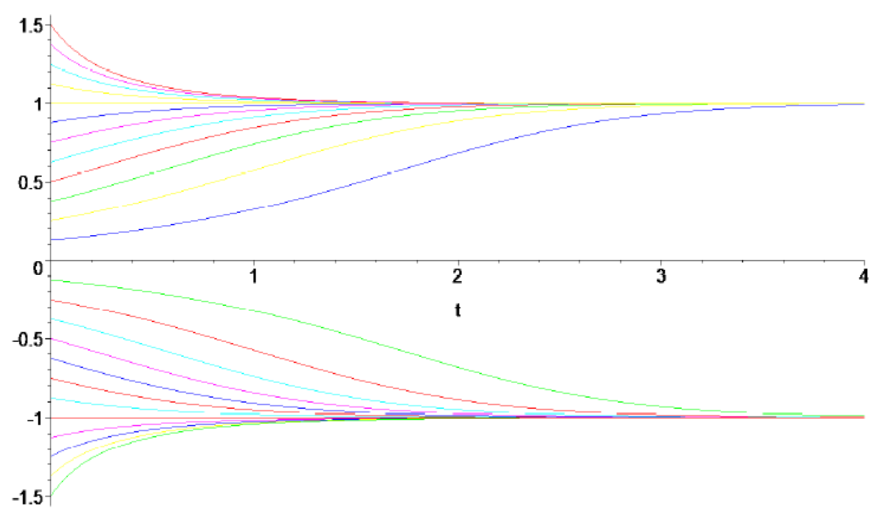



```
> dxdt:=a^2*x-x^3;
```
$$dxdt := a^2x - x^3$$
```
> equi:=[solve(dxdt,x)];
```
$$equi := [0, -a, a]$$
```
> dd:=diff(dxdt,x); map(v->subs(x=v,dd),equi);
```
$$dd := a^2 - 3x^2$$
$$[a^2, -2a^2, -2a^2]$$
```
> plot(subs(a=1,a^2*x-x^3),x=-1.2..1.2);
```



This D.E. can be solved exactly.
Here is what a few solutions look like.

```
> eq:=diff(x(t),t)=subs(x=x(t),dxdt);
```
$$eq := \frac{d}{dt}x(t) = dxdt$$
```
> [seq(-3*a/2+k*3*a/24,k=0..24)]:
  map(v->subs(dsolve({x(0)=v,eq},x(t)),x(t)),%):
  subs(a=1,%): plot(convert(%,set),t=0..4);
```



5. Solve the Torricelli differential equation $dh/dt = -\sqrt{h}$ with initial condition $h(0) = 2$. Sketch the solution and describe its long-term behavior.

(5) $\dfrac{dh}{dt} = -\sqrt{h}$ $\qquad h(0) = 2$

$$\int \frac{dh}{\sqrt{h}} = -\int dt = -t + C$$

$$\int \frac{dh}{\sqrt{h}} = \int h^{-\frac{1}{2}}\,dh = \frac{h^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2h^{\frac{1}{2}} = 2\sqrt{h}$$

$$2\sqrt{h} = -t + C$$

Plug in: $t = 0$, $h = 2$

$$2\sqrt{2} = C$$

$$\therefore \; 2\sqrt{h} = -t + 2\sqrt{2}$$

$$\boxed{h = \left(-\frac{t}{2} + \sqrt{2}\right)^2}$$

Long run: By $t = 2\sqrt{2}$ h will become zero.

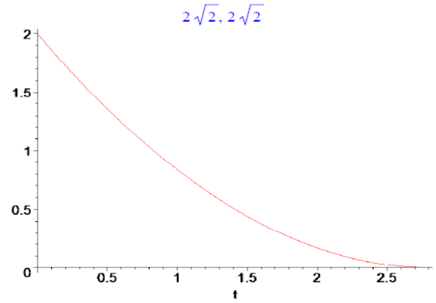
```
> diff(h(t),t)=-sqrt(h(t)); dsolve({%,h(0)=2},h(t)): allvalues(%):
  hh:=subs(%,h(t));
```

$$\frac{d}{dt}\mathrm{h}(t) = -\sqrt{\mathrm{h}(t)}$$

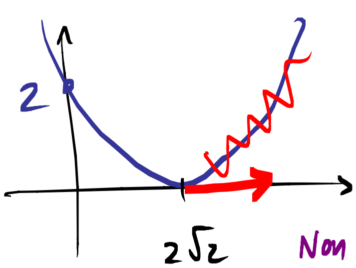
$$hh := \frac{1}{4}t^2 - t\sqrt{2} + 2$$

```
> solve(hh,t); plot(hh,t=0..%[1]);
```

$$2\sqrt{2},\, 2\sqrt{2}$$



Question: What about after that?



Non-uniqueness!

$$h = \begin{cases} \left(-\frac{t}{2} + \sqrt{2}\right)^2 & \text{if } 0 \le t \le 2\sqrt{2} \\ 0 & \text{if } t > 2\sqrt{2} \end{cases}$$

is a solution

Furthermore, since most real-life scenarios involve damping, this is the most likely outcome.