CS 3333.002 Final Exam Sp'16

① $\underline{\gcd}(a, m)$ is a common divisor of $\underline{a}$ & $\underline{m}$.

Since $a \equiv b \bmod m$ ∃ k $b = \underline{a} + k\underline{m}$ ∴ $\gcd(a,m) \mid b$

so $\gcd(a,m)$ is a Common divisor of b and m.

∴ $\gcd(a,m) \mid \gcd(b,m)$

Similarly $\gcd(b,m) \mid \gcd(a,m)$ ∴ $\gcd(a,m) = \gcd(b,m)$ ☺

Alt: Can show integer combinations of a, m are the same as integer comb. of b and m.

E.g. $sb + tm = s(a + km) + tm = sa + (sk+t)m$

If we take all those > 0, then we have the $\underline{\text{same}}$ min (gcd). ☺

The converse does not hold, for example let $m = 3$, $a = 2$, $b = 10$.

Then $\gcd(a,m) = 1 = \gcd(b,m)$

But $a \not\equiv b \bmod m$ $a - b = -8$ is not div. by 3 ☹

② $313 = 2\cdot 131 + 51$

$131 = 2\cdot 51 + 29$

$51 = 29 + 22$

$29 = 22 + 7$

$22 = 3\cdot 7 + \boxed{1}$ gcd

$1 = 22 - 3\cdot 7 = 22 - 3(29 - 22) = -3\cdot 29 + 4\cdot 22 =$

$= -3\cdot 29 + 4(51 - 29) = 4\cdot 51 - 7\cdot 29 =$

$= 4\cdot 51 - 7(131 - 2\cdot 51) = -7\cdot 131 + 18\cdot 51$

$= -7\cdot 131 + 18(313 - 2\cdot 131) = 18\cdot 313 - 43\cdot 131$

∴ $131^{-1} = -43 \bmod 313 = \underline{270 \bmod 313}$

③
$$\begin{aligned} x &\equiv 2 \bmod 5 \\ x &\equiv 5 \bmod 6 \\ x &\equiv \underset{\substack{\uparrow \\ b_i}}{3} \bmod \underset{\substack{\uparrow \\ m_i}}{7} \end{aligned}$$

$M = m_1 \cdot m_2 \cdot m_3 = 210$

m_i	M/m_i	$\bmod m_i$	$\frac{M}{m_i}^{-1} \bmod m_i$	b_i
5	42	2	3	2
6	35	5	5	5
7	30	2	4	3

$42\cdot 3\cdot 2 + 35\cdot 5\cdot 5 + 30\cdot 4\cdot 3 = 1487 \equiv \boxed{17 \bmod 210}$

④ Basis: $n=2$ $\quad 1+\frac{1}{4}=1.25<2-\frac{1}{2}=1.5$ ☺

Let $n>2$. $\quad 1+\frac{1}{4}+\dots+\frac{1}{(n-1)^2}+\frac{1}{n^2}<2-\frac{1}{n-1}+\frac{1}{n^2}=2-\frac{n-1}{n^2}\leq 2-\frac{1}{n^2}$

Since $n\geq 2$, $n-1\geq 1$, $\frac{n-1}{n^2}\geq\frac{1}{n^2}$, $-\frac{n-1}{n^2}\leq-\frac{1}{n^2}$

⑤ $x_n=2x_{n-1}+15x_{n-2}$. Let $x_n=r^n$. Then $r^n=2r^{n-1}+15r^{n-2}$.

$\therefore r^2-2r-15=0 \quad \therefore r=5,-3. \quad \therefore x_n=a\cdot 5^n+b(-3)^n$

$x_0=1,\ x_1=2 \Rightarrow a+b=1,\ 5a-3b=2$

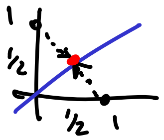
$$\left[\begin{array}{cc:c}1&1&1\\5&-3&2\end{array}\right]\to\begin{bmatrix}1&1&1\\0&-8&-3\end{bmatrix}\to\begin{bmatrix}1&1&1\\0&1&3/8\end{bmatrix}\to\begin{bmatrix}1&0&5/8\\0&1&3/8\end{bmatrix}\quad\therefore a=\frac{5}{8},\ b=\frac{3}{8}$$

$$\therefore x_n=\frac{5}{8}5^n+\frac{3}{8}(-3)^n=\underline{\frac{1}{8}\left[5^{n+1}-(-3)^{n+1}\right]}$$

⑥
$$\left[\begin{array}{ccc:c}0&1&1&a\\1&0&1&b\\1&1&0&c\end{array}\right]\to\begin{bmatrix}1&1&0&c\\0&1&1&a\\1&0&1&b\end{bmatrix}\to\begin{bmatrix}1&1&0&c\\0&1&1&a\\0&-1&1&b-c\end{bmatrix}\to\begin{bmatrix}1&1&0&c\\0&1&1&a\\0&0&2&a+b-c\end{bmatrix}\to$$

$$\begin{bmatrix}1&1&0&c\\0&1&1&a\\0&0&1&\frac{a+b-c}{2}\end{bmatrix}\to\begin{bmatrix}1&1&0&c\\0&1&0&\frac{a-b+c}{2}\\0&0&1&\frac{a+b-c}{2}\end{bmatrix}\to\left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\end{array}\ \boxed{\begin{array}{c}\frac{-a+b+c}{2}\\\frac{a-b+c}{2}\\\frac{a+b-c}{2}\end{array}}\right]\begin{array}{l}\leftarrow x\\\leftarrow y\\\leftarrow z\end{array}$$

⑦ Projection:



$\begin{bmatrix}1\\0\end{bmatrix}\mapsto\begin{bmatrix}1/2\\1/2\end{bmatrix}$
$\begin{bmatrix}0\\1\end{bmatrix}\mapsto\begin{bmatrix}1/2\\1/2\end{bmatrix}$
$\therefore A=\begin{bmatrix}1/2&1/2\\1/2&1/2\end{bmatrix}$, If $\bar{x}\in$ Image(A) $A\bar{x}=\bar{x}$, so $A^2=A$, so $A^4=A$.

Rotation: $\begin{bmatrix}1\\0\end{bmatrix}\mapsto\begin{bmatrix}0\\1\end{bmatrix},\ \begin{bmatrix}0\\1\end{bmatrix}\mapsto\begin{bmatrix}-1\\0\end{bmatrix}\quad\therefore A=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$. $A^4=$ rot by $360°$ so $A^4=I$.

⑧ $A=\begin{bmatrix}-9&8\\-12&11\end{bmatrix}\quad\det(A-\lambda I)=\det\begin{bmatrix}-9-\lambda&8\\-12&11-\lambda\end{bmatrix}=\lambda^2-2\lambda-3=(\lambda-3)(\lambda+1)$

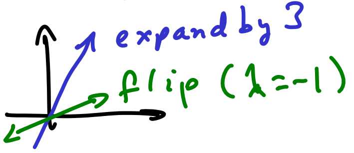
$\lambda=3,-1$

$\operatorname{rref}(A-3I)=\operatorname{rref}\begin{bmatrix}-12&8\\-12&8\end{bmatrix}=\begin{bmatrix}1&-2/3\\0&0\end{bmatrix}\qquad x-\frac{2}{3}y=0\qquad v_1=\begin{bmatrix}2\\3\end{bmatrix}$

eigenvectors

$\operatorname{rref}(A+I)=\operatorname{rref}\begin{bmatrix}-8&8\\-12&12\end{bmatrix}=\begin{bmatrix}1&-1\\0&0\end{bmatrix}\qquad x-y=0\qquad v_2=\begin{bmatrix}1\\1\end{bmatrix}$

Let $S=\begin{bmatrix}2&1\\3&1\end{bmatrix}$, then $S^{-1}=\begin{bmatrix}-1&1\\3&-2\end{bmatrix}$, $A\cdot S=\begin{bmatrix}6&-1\\9&-1\end{bmatrix}$, $\underline{S^{-1}AS=\begin{bmatrix}3&0\\0&-1\end{bmatrix}}$



⑨ Full house: $\frac{13\cdot C(4,3)\cdot 12\cdot C(4,2)}{C(52,5)} = \boxed{\frac{6}{4165} = 0.00144}$ $\qquad C(n,k)=\frac{n!}{k!(n-k)!}$

Two pair: $\frac{C(13,2)\,C(4,2)\,C(4,2)\cdot \underset{\overset{\uparrow}{\text{5th card}}}{11\cdot 4}}{C(52,5)} = \boxed{\frac{198}{4165} \approx 0.047539}$

⑩ $\int_{-\infty}^{\infty} p(t)\,dt = \int_0^{20}(mt+0.1)\,dt = \left[\frac{mt^2}{2}+0.1t\right]_0^{20} = \frac{m20^2}{2}+2$

Set $\frac{m20^2}{2}+2=1$, then $m=-\frac{1}{200}=\boxed{-0.005}$

$$\mu=\int_{-\infty}^{\infty} t\,\varphi(t)\,dt = \int_0^{20}(0.1t-0.005t^2)\,dt = \left[0.1\frac{t^2}{2}-0.005\frac{t^3}{3}\right]_0^{20}$$

$$= 0.1\frac{20^2}{2}-0.005\frac{20^3}{3} = \boxed{\frac{20}{3}\approx 6.67}$$

$$\text{cdf}(t) = \begin{cases} 0 & \text{for } t<0 \\ -0.005\frac{t^2}{2}+0.1t & \text{for } 0\le t\le 20 \\ 1 & \text{for } t>20 \end{cases}$$

$\text{cdf}(t)=\frac{1}{2} \Rightarrow 0\le t\le 20$ & $-0.005\frac{t^2}{2}+0.1t=\frac{1}{2}$

$t^2-40t+200=0 \Rightarrow t=20\pm 10\sqrt{2}\approx \boxed{5.86}, \cancel{34.14}$