Name: ______________________________

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Choose 8 questions to answer. Enter the selected problems in the top parts of the boxes above. Please supply brief narration with your formulas, state the results you use, and show all work!

1. Suppose f is holomorphic on a domain. Prove that f is constant
   (a) if the real part $\Re[f]$ is constant.
   (b) if the modulus $|f|$ is constant.

(a) Write $f(x+iy) = u(x,y) + iv(x,y)$. Since $u = \Re[f] = \text{const}$, $u_x = u_y = 0$. By the Cauchy-Riemann equations $v_y = u_x = 0$ and $v_x = -u_y = 0$, so $v = \text{const}$.

(b) Write $f(x+iy) = \rho(x,y)e^{i\psi(x,y)}$. If $\rho = 0$, we are done, so assume $\rho \neq 0$. Since $\rho = |f| = \text{const}$, $\rho_x = \rho_y = 0$. By the Cauchy-Riemann equations $\psi_y = \rho_x/\rho = 0$ and $\psi_x = -\rho_y/\rho = 0$, so $\psi = \text{const}$.

2. Evaluate the following integrals around the unit circle: (a) $\displaystyle\int \frac{dz}{\sin z}$ (b) $\displaystyle\int \frac{dz}{z^3 - 2iz^2}$

(a) The only zero of $\sin z$ inside the unit circle is $z = 0$. Expand the integrand in a Laurent series at $z = 0$. Since $\sin z = z - z^3/3! + \ldots$, long division gives $1/\sin z = 1/z + \ldots$
The residue is 1, so the integral is $2\pi i$.

(b) Factor the denominator: $z^3 - 2iz^2 = z^2(z - 2i)$. The only zero of the denominator inside the unit circle is $z = 0$. Let $f(z) = (z-2i)^{-1}$. Then $f'(z) = -(z-2i)^{-2}$. By Cauchy's integral formula, the integral is $2\pi i f'(0) = -2\pi i(-2i)^{-2} = \pi i/2$.

3. Expand $f(z) = \dfrac{1}{z^2 + 3iz - 2}$ in a Laurent series valid in the annulus $\{z \in \mathbf{C} \colon 1 < |z| < 2\}$.

$$
\begin{aligned}
\frac{1}{z^2+3iz-2} &= \frac{1}{(z+i)(z+2i)} = -\frac{i}{z+i} + \frac{i}{z+2i} = -\frac{i}{z}\cdot\frac{1}{1+\frac{i}{z}} + \frac{1}{2}\cdot\frac{1}{1+\frac{z}{2i}} \\
&= -\frac{i}{z}\sum_{k=0}^{\infty}\left(-\frac{i}{z}\right)^k + \frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{z}{2i}\right)^k = \sum_{k=0}^{\infty}(-i)^{k+1}z^{-k-1} + \sum_{k=0}^{\infty} i^k 2^{-k-1} z^k \\
&= \sum_{k=-\infty}^{-1}(-i)^{-k}z^k + \sum_{k=0}^{\infty} i^k 2^{-k-1}z^k = \sum_{k=-\infty}^{\infty} c_k z^k \text{ where } c_k = \begin{cases} i^k 2^{-k-1} & \text{if } k \geq 0 \\ (-i)^{-k} & \text{if } k < 0 \end{cases}
\end{aligned}
$$

Note that the subseries with $k < 0$ converges for $|z| > 1$ and the subseries with $k \geq 0$ for $|z| < 2$. Therefore, the series converges in the given annulus.

4. Construct a Möbius transformation taking the real axis to the unit circle. Prove that the transformation you constructed does what is claimed. You may use various properties of Möbius transformations in your proof.

Pick three points on the real axis, say $[0, 1, \infty]$, and three on the unit circle, say $[-i, 1, i]$. Define a Möbius transformation T taking the last three to the first:

$$T(z) = \frac{z+i}{z-i} \cdot \frac{1-i}{1+i} = \frac{-iz+1}{z-i}$$

and compute its inverse

$$T^{-1}(z) = \frac{z-i}{-iz+1}$$

It is easily checked that $T^{-1}(0) = -i$, $T^{-1}(1) = 1$, $T^{-1}(\infty) = i$. Since Möbius transformations preserve generalized circles (circles or straight lines), T^{-1} takes the real axis to a circle containing $[-i, 1, i]$. Generalized circles are uniquely determined by 3 points, so the circle in question must be the unit circle.

5. Suppose $f \not\equiv 0$ has the Laurent expansion at the origin $\sum_{k=n}^{\infty} a_k z^k$, where $n \in \mathbf{Z}$. Prove that

   (a) $\exists$ punctured disc D^* around 0 containing neither singularities nor zeros of f.
   (b) $2\pi i\, a_{-1}$ is the integral of f along a circle in the interior of D^* around 0.

(a) Since $f \not\equiv 0$, $\exists k\ a_k \neq 0$. Without loss of generality we may assume that $a_n \neq 0$. Then

$$f(z) = z^n \sum_{k=n}^{\infty} a_k z^{k-n} = z^n g(z) \text{ where } g(z) = \sum_{k=0}^{\infty} a_{k+n} z^k$$

Since g is analytic at 0, g is continuous at 0. Since $g(0) = a_n \neq 0$, by continuity $g(z)$ is nonzero in some disc D around 0.

Here is a more detailed argument (optional): since $|g(0)| > 0$, continuity says that $\exists \delta > 0\ |z| < \delta \Rightarrow |g(z) - g(0)| < |g(0)|$. By the triangle inequality $|g(0)| = |g(0) - g(z) + g(z)| \leq |g(0) - g(z)| + |g(z)|$. Thus, if $|z| < \delta$, $|g(0)| < |g(0)| + |g(z)|$, so $|g(z)| > 0$, so $g(z) \neq 0$.

Thus, the only possible singularity or zero (depending on the sign of n) of f in D is 0.

(b) The radius of convergence of a Laurent series is the distance from the center of expansion to the nearest singularity. By part (a) the Laurent series for f at 0 converges in D^*. Furthermore, the convergence is uniform on any compact subset of D^*, in particular on the contour of integration. Therefore we may interchange summation and integration:

$$\oint f(z)\, dz = \oint \left[\sum_{k=n}^{\infty} a_k z^k \right] dz = \sum_{k=n}^{\infty} a_k \oint z^k\, dz$$

Since the integral of z^k is 0 except when $k = -1$, all but one term in the series vanish. The remaining integral can be taken directly. Let $z = re^{i\theta}$, where r is the radius of the circle of integration. Then $dz = re^{i\theta} i\, d\theta = iz\, d\theta$, so $z^{-1}\, dz = i\, d\theta$ and

$$\oint f(z)\, dz = a_{-1} \oint z^{-1}\, dz = a_{-1} i \int_{-\pi}^{\pi} d\theta = a_{-1} i\, 2\pi$$

6. Suppose f is entire. Prove that $M(r) = \max_{|z|=r} |f(z)|$ is a nondecreasing function of r.

By the maximum modulus principle $M(r) = \max_{|z|\le r} |f(z)|$.

7. Let $r > 0$. Derive Cauchy's inequalities for a function f analytic at z_0

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|$$

Let γ be a circle of sufficiently small radius r around z_0 inside the domain of analyticity of f. By Cauchy's integral formula

$$\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Let $M = \max_{|z|=r} |f(z)|$. By the triangle inequality for integrals (the ML estimate)

$$\frac{2\pi}{n!}\left|f^{(n)}(z_0)\right| = \left|\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \le \oint_\gamma \frac{|f(z)|}{|z-z_0|^{n+1}}\,|dz| \quad \le \frac{M}{r^{n+1}} \oint_\gamma |dz| = \frac{M}{r^{n+1}}\,2\pi r = \frac{2\pi M}{r^n}$$

8. Let $M, p > 0$. Suppose $f(z)$ is entire and $|f(z)| \le M\,|z|^p$ for all z with $|z|$ sufficiently large. Prove that f is a polynomial with $\deg f \le p$. You may use Cauchy's inequalities.

Since f is entire, we may choose arbitrary r in Cauchy's inequalities (with $z_0 = 0$):

$$\left|f^{(n)}(0)\right| \le \frac{n!}{r^n} \max_{|z|=r} |f(z)| \le \frac{n!}{r^n} \max_{|z|=r} M\,|z|^p = \frac{n!}{r^n} M r^p = \frac{n!}{M} r^{p-n}$$

Taking limit of both sides as $r \to \infty$ show that the Maclaurin coefficients of f are 0 for $n > p$.

9. Let D denote the unit disc. Suppose $g\colon D \to D$ is holomorphic with $g(0) = 0$.

   (a) Show that $h(z) = g(z)/z$ has a removable singularity at 0.

   (b) Prove that $|g(z)| \le |z|$ and $|g'(0)| \le 1$.

   (a) Expand g at 0 in a Maclaurin series $g(z) = \sum_{n=0}^{\infty} a_n z^n$. Note that since $g(0) = 0$, $a_0 = 0$.

   Since g is holomorphic on D, the series converges on D, so on D^*

$$h(z) = g(z)/z = \left(\sum_{n=1}^{\infty} a_n z^n\right)/z = \sum_{n=1}^{\infty} a_n z^{n-1} = \sum_{n=0}^{\infty} a_{n+1} z^n$$

   The series by gives a holomorphic function h^* on D that agrees with h on D^*.

   (b) Since $|g(0)| = 0$, $|g(z)| \le |z|$ is satisfied trivially at $z = 0$. Since the image of h is contained in D, for $z \ne 0$ we have $|g(z)|/|z| = |g(z)/z| = |h(z)| < 1$, so $|g(z)| < |z|$.

   Since $|h^*(z)| = |h(z)| < 1$ on D^*, by continuity $|g'(0)| = |a_1| = |h^*(0)| \le 1$. In fact, by the maximum modulus principle $|g'(0)| < 1$.

10. State and prove the Riemann extension theorem on removable singularities.

11. State and prove the principle of analytic continuation.

12. State and prove the theorem of Weierstrass on convergent sequences of analytic functions.