Final exam / 2004.5.7 / Algebra II / MAT 5313

Name: ____

Please show all work and justify your answers.

Let K denote a commutative ring.

1. Suppose F is free K-module on 3 generators. Prove from first principles that the set $\operatorname{Alt}_2 F$ of all bilinear alternating maps $F^2 \to K$ is a free cyclic K-module.

2. Let A be a K-module and define $f\colon A \times K^2 \to A^2$ by $f(a,(\kappa,\lambda)) = (\kappa a, \lambda a)$.
   (a) Prove that f is bilinear.
   (b) Prove that f is universal among bilinear maps on $A \times K^2$ by showing that for any bilinear $g\colon A \times K^2 \to C$ there exists unique linear $g'\colon A^2 \to C$ with $g = g' \circ f$.
   Hint: Write $(a,b) = (a,0) + (0,b)$. What elements does f take to the two summands?

3. Prove that
   (a) $\mathbf{Z}_2 \otimes_{\mathbf{Z}} \mathbf{Z}_5 \cong 0$
   (b) $\mathbf{Z}^2 \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}^2$

4. Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ is a short exact sequence of K-modules and C is free. Prove that $B \cong A \oplus C$.

5. Let $K = \mathbf{Z}[x]$. Let $I = \{p \in K\colon p(0) \text{ is even}\}$. Prove that
   (a) I is an ideal of K
   (b) I is not a free K-module

6. (a) Prove that the symmetric group S_3 is solvable.
   (b) Prove that S_3 is not nilpotent.

7. Suppose G is a group of order n such that for each prime divisor p of n there is only one Sylow p-group in G. What can you conclude about the structure of G?

8. Suppose F is a field of characterstic ∞ and $u \in F$ satisfies $u^2 + 3u - 1 = 0$. Let $s = \dfrac{1}{1-u}$.
   (a) Express s as a linear combination of 1 and u.
   (b) Find a polynomial with rational coefficients satisfied by s.

1	2	3	4	5	6	7	8	total (80)
								%