Midterm 2 / 2003.4.23 / Theory of Functions of a Complex Variable II / MAT 5233.001

Name: ____

Please show all work. If you use a known result in your proof, state the result in full.

1. (10 pts.) Suppose f is entire. What are all the possibilities for the range of e^f? Prove your assertion.

2. (10 pts.) Suppose

   (i) $\Omega \subseteq \mathbf{C}$ is a domain (open and connected subset of $\mathbf{C}$) and

   (ii) $\lambda\colon \Omega \to \mathbf{R}$ is twice differentiable and satisfies $\Delta^c(\ln \lambda) \geq \lambda$
   (recall that Δ^c is defined as one fourth of the Laplacian Δ).

   Show that if $f\colon D^* \to \Omega$ is analytic on the punctured unit disc $D^* = \{z \in \mathbf{C}\colon 0 < |z| < 1\}$, then
   $$\left|f'(z)\right|^2 \lambda(f(z)) \leq \frac{1}{2\left(|z| \ln |z|\right)^2}, \quad \text{for } z \in D^*.$$

   [You may use Ex. 222.1 which is a consequence of Ahlfors's version of the Schwartz lemma saying that if $g\colon H \to \Omega$ is analytic on the upper half plane $H = \{z \in \mathbf{C}\colon \Im[z] > 0\}$, then
   $$\left|g'(z)\right|^2 \lambda(g(z)) \leq \frac{1}{2\left(\Im[z]\right)^2}, \quad \text{for } z \in H.$$

   Hint: modify the exponential function to map H to D^*.]

3. (10 pts.) Let $H = \{z \in \mathbf{C}\colon \Im[z] > 0\}$ denote the upper half plane.
   Define $g(x) = 1$ for $x > 0$ and $g(x) = 2$ for $x < 0$.

   (a) Find a harmonic $\Phi\colon H \to \mathbf{R}$ such that $\Phi(x, 0) = g(x)$.

   (b) Find a harmonic $\Psi\colon H \to \mathbf{R}$ such that $F(x + iy) = \Phi(x, y) + i\Psi(x, y)$ is analytic.

   (c) Sketch a few curves of constant Φ and Ψ (so-called equipotential and flow lines).

Cauchy-Riemann equations:
$f(x+iy) = u + iv$: $\quad u_x = v_y, \quad v_x = -u_y$
$f(re^{i\theta}) = u + iv$: $\quad ru_r = v_\theta, \quad rv_r = -u_\theta$
$f(x+iy) = \rho e^{i\psi}$: $\quad \rho_x = \rho\psi_y, \quad \rho_y = -\rho\psi_x$
$f(re^{i\theta}) = \rho e^{i\psi}$: $\quad r\rho_r = \rho\psi_\theta, \quad \rho_\theta = -r\rho\psi_r$

1	2	3	total (30)
			%

Prelim. course grade: %