Final exam / 2003.5.6 / Theory of Functions of a Complex Variable II / MAT 5233.001

Name: ______________________________

Please show all work and justify answers. If you use a known result, state the result in full.

1. (10 pts.) Suppose Ω is a domain in the complex plane and f_n is a sequence in $\mathcal{H}(\Omega)$ such that $f_n \to f$ uniformly on compact subsets of Ω, The Weierstrass theorem says that in this case $f \in \mathcal{H}(\Omega)$. Prove the second part of the theorem which says that for any k the k-th derivatives $f_n^{(k)} \to f^{(k)}$ uniformly on compact subsets of Ω.

2. (10 pts.) Find the first three nontrivial terms of the Laurent series at the origin of

$$\text{(a) } f(z) = \frac{\cos z}{z - z^3} \qquad \text{(b) } f(z) = \cot z$$

3. (10 pts.) For each of the following covering maps p, how many elements are there in each stalk $p^{-1}(x)$? Compute and sketch $p^{-1}(1)$. Illustrate that p is indeed a covering map by sketching an evenly covered neighborhood of 1 and its preimage under p.

$$\text{(a) } p(z) = z^4 \colon \mathbf{C} \setminus \{0\} \to \mathbf{C} \setminus \{0\} \qquad \text{(b) } p(z) = \sin z \colon \mathbf{C} \to \mathbf{C}$$

4. (10 pts.) Suppose f is entire and never zero. What are all the possibilities for the range of $1/f$? Prove your assertion.

5. (10 pts.) Suppose λ is a real valued twice differentiable function on a domain $\Omega \subseteq \mathbf{C}$ such that $\frac{1}{4}\Delta(\ln \lambda) \geq \lambda$. Show that if $f : H \to \Omega$ is analytic on the right half plane $H = \{z \in \mathbf{C}\colon \Re[z] > 0\}$, then

$$|f'(z)|^2 \lambda(f(z)) \leq \frac{1}{2(\Re[z])^2} \quad \text{for } z \in H.$$

[You may use Ahlfors's version of the Schwartz lemma saying that if $g\colon D \to \Omega$ is analytic on the unit disc $D = \{z \in \mathbf{C}\colon\ |z| < 1\}$, then

$$|g'(z)|^2 \lambda(g(z)) \leq \frac{2}{\left[1 - |z|^2\right]^2} \quad \text{for } z \in D.$$

Hint: Find an analytic map $h\colon D \to H$ and apply the lemma to the composition $g = f \circ h$.]

6. (10 pts.) Let Q be the positive quadrant of the complex plane.
   (a) Find a harmonic Φ on Q that approaches 1 on the positive x axis and -1 on the positive y axis.
   (b) Find a harmonic Ψ on Q such that $F = \Phi + i\Psi$ is analytic on Q.
   (c) Sketch a few curves of constant Φ and Ψ (so-called equipotential and flow lines).

Cauchy-Riemann equations:

$$f(x+iy) = u + iv\colon \quad u_x = v_y, \quad v_x = -u_y$$
$$f(re^{i\theta}) = u + iv\colon \quad ru_r = v_\theta, \quad rv_r = -u_\theta$$
$$f(x+iy) = \rho e^{i\psi}\colon \quad \rho_x = \rho\psi_y, \quad \rho_y = -\rho\psi_x$$
$$f(re^{i\theta}) = \rho e^{i\psi}\colon \quad r\rho_r = \rho\psi_\theta, \quad \rho_\theta = -r\rho\psi_r$$

1	2	3	4	5	6	total (60)
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