Theory of Functions of a Complex Variable I, MAT 5223
Final, due 10:45pm Wednesday December 14, 1994
Instructor: D. Gokhman

Name: ______________________________

1	2	3	4	5	6	total (140)

1. (10 pts.) ROOTS
   Find all solutions of the equation $z^5 = -4 + 3i$ and plot them.

2. (10 pts.) STEREOGRAPHIC PROJECTION
   Given two points $z_1, z_2 \in \mathbf{C}$, find the length of the chord between the corresponding points on the Riemann sphere? Pick two specific values for z_1 and z_2 and sketch. For which z_1, z_2 are the two corresponding points on the sphere diametrically opposite?

3. (10 pts.) EULER'S FORMULA
   For $a, b \in \mathbf{R}$ find the partial sum: $$\sum_{k=0}^{n} \sin(a + kb)$$

4. (40 pts.) TRANSFORMATIONS
   (a) Sketch the image of the segment $\operatorname{Re} z = \operatorname{Im} z$ between 0 and $1 + i$ under $w = e^z$? What is the arclength of the image?
   (b) Show that the group of Möbius transformations is not commutative.
   (c) For which $a \in \mathbf{R}$ is the group generated by $w = e^{ia} z$ finite?
   (d) Find the group of transformations corresponding to rotations of the Riemann sphere with respect to the imaginary axis.

5. (40 pts.) INTEGRATION
   (a) Find $\int z^n \log(z)\, dz$, where the path of integration is the unit circle (counterclockwise).
   (b) Find $\int \frac{e^{2z}}{z^3}\, dz$, where the path of integration is the unit circle (counterclockwise).
   (c) Suppose f/z is continuous in a sector centered at the origin with aperture θ. Let the path $\gamma(r)$ be the intersection of $|z| = r$ with the sector. Show that $\int_{\gamma(r)} \frac{f(z)}{z}\, dz \to \theta i f(0)$ as $r \to 0$.
   (d) Suppose g is entire and $z\, g(z) \to 0$ as $z \to \infty$. Show that integrals of $g(z)\, dz$ along any two rays from 0 to ∞ are equal, assuming they exist. Hint: use the results of part (c).

6. (30 pts.) POWER SERIES
Find the Maclaurin series for each of the following funcions and determine its radius of convergence

   (a) $\dfrac{z}{z+2}$
   (b) $\tan(z)$
   (c) $\displaystyle\int_0^z \frac{\sin(z)}{z}\, dz$

These problems are from *A collection of problems on complex analysis* by Volkovyskii, Lunts and Aramanovich, 1960 (Dover 0486669130, QA331.7.V6513 1991)