Final / 2013.12.17 / MAT 5173.001 / Algebra I

Name: ______________________________

Please show all work and justify your answers. Throughout $G$ denotes a group, $R$ a commutative ring with unity, $D$ an integral domain, and $F$ a field.

1. Show that the set of all inner automorphisms of $G$ (maps of the form $x \mapsto gxg^{-1}$ where $g \in G$) is a normal subgroup of the group of all automorphisms of $G$ under composition.
2. Suppose $\varphi\colon G \to H$ is a surjective group homomorphism and $T < H$. Prove that $T \triangleleft H$ if and only if $\varphi^* T \triangleleft G$. Show that in that case $G/\varphi^* T \cong H/T$.
3. Suppose $f\colon A \to B$ is an $R$-module morphism. Prove that the natural inclusion $i\colon \ker f \to A$ is universal among $R$-module morphisms $g$ into $A$ such that $f \circ g = 0$. Specify the functor and the corresponding universal element.
4. Dualize the previous problem by proving that the natural projection $\pi\colon B \to B/f_*(A)$ is universal among $R$-module morphisms $g$ from $B$ such that $g \circ f = 0$. Specify the functor and the corresponding universal element.
5. Prove that the natural inclusion $i\colon D \to Q(D)$ of an integral domain in its field of quotients is universal among injective ring morphisms from an integral domain $D$ to fields. Specify the functor and the corresponding universal element.
6. Prove or disprove that the ring $\mathbf{R}^{\mathbf{R}}$ of all functions $\mathbf{R} \to \mathbf{R}$ with pointwise addition and multiplication is an integral domain. What are the units of $\mathbf{R}^{\mathbf{R}}$? Let $c \in \mathbf{R}$. Prove that $J = \{f\colon \mathbf{R} \to \mathbf{R}\colon f(c) = 0\}$ is a maximal ideal of $\mathbf{R}^{\mathbf{R}}$.
7. Show that for any $R$-module $M$, there exists a surjective $R$-module morphism from some free $R$-module $F$ to $M$.
8. Prove that coproduct of a nonempty family of free $R$-modules is free.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | total (80) | % |
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