Final exam / 2023.5.9 / MAT 4273.001 / Topology

Name: ______________________________

Please show all work and justify your answers.

1. Suppose A_i is a sequence of subsets of a topological space X

   (a) Show that $\bigcup_{i=1}^{\infty} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}$, where bar denotes closure ($\overline{A} = \mathrm{cl}(A)$)

   (b) Illustrate with a concrete example that equality does not always hold.

2. Suppose X is a topological space, $f\colon X \to \mathbf{R}$ is a continuous real-valued function, and $S \subseteq X$ such that $\overline{S} = X$ (subsets whose closure is the whole space are called dense).

   (a) If $f(x) = 0$ for all $x \in S$, prove that $f(x) = 0$ for all $x \in X$

   (b) Use (a) to prove that if two continuous real-valued functions on X agree on S, then they agree on X (hint: consider the difference).

3. Let $X = [-1, 0) \cup (0, 1] \subseteq \mathbf{R}$ with the usual Euclidean metric. Explain why X is not compact and then prove it directly by exhibiting an open cover of X that has no finite subcover.

4. Explain why X in the preceding problem is not connected and then prove it directly by exhibiting a separation of X (see Definition 54.3, Kasriel p.109).

5. Exercise 57.3 (Kasriel p.115)

6. True/false — circle your choice (no justification necessary).

T F (a) The union of a family of closed subsets of the Euclidean plane is not open.

T F (b) A subset of $\mathbf{R}^n$ is compact if and only if it is close and bounded.

T F (c) A continuous integer-valued function on a connected space is constant.

T F (d) A real polynomial of odd degree must have at least one real root.

T F (e) The union of two polygonally connected subsets of $\mathbf{R}^n$ is polygonally connected.

7. Bonus: exercises 55.6–7 (Kasriel p.113)