Final exam / 2004.12.10 / Number Theory / MAT 4253.001

Name: ____________________

Please show all work and justify your statements.

1. Prove that equations $\gcd(x, y) = g$ and $xy = b$ have simultaneous solutions over positive integers if, and only if, $g^2|b$.

2. Prove that if a and b are positive integers, then $\gcd(a, a + b)|b$.

3. Prove that if a and b are positive integers such that $\gcd(a, b) = \mathrm{lcm}(a, b)$, then $a = b$.

4. Find all integer solutions of $20x \equiv 8 \bmod 30$.

5. Find all integer solutions of $15x + 21y = 6$.

6. Solve the system of congruences $x \equiv 2 \bmod 5$, $x \equiv 5 \bmod 7$, $x \equiv 3 \bmod 8$.

7. Solve $x^3 + x^2 \equiv 5 \bmod 343$.

8. Solve the recurrence $u_n = 3u_{n-1} - 2u_{n-2}$ subject to initial conditions $u_0 = a$, $u_1 = b$.

9. Prove that if n is a positive integer, then $\sum_{d|n} d\,\mu(d) = (-1)^{\omega(n)}\varphi(n)s(n)/n$,

   where $s(n)$ is the largest square free divisor of n, i.e. $s(n) = \prod_{p|n} p$.

   You may use the fact that $\varphi(n) = n \prod_{p|n}(1 - 1/p)$.

1	2	3	4	5	6	7	8	9	total (90)	%