Final exam / 2019.12.9 / MAT 4233.001 / Modern Abstract Algebra

Name: ____

Please show all work. If you use a theorem, name it or state it.

1. Suppose x is an element of a finite group G. Show that
   (a) x has finite order (denote it k),
   (b) $x^n = e$ if and only if k divides n,
   (c) $x^{|G|} = e$.

2. Sketch the subgroup lattice for $\mathbf{Z}_{12}$. For each subgroup, list all the elements and indicate all possible generators of the subgroup.

3. Define a function from the integers to the multiplicative group of nonzero complex numbers $\varphi\colon \mathbf{Z} \to \mathbf{C}^*$ by $\varphi(k) = e^{2k\pi i/5}$.
   (a) Prove that φ is a group homomorphism.
   (b) What subgroup of $\mathbf{Z}$ is the kernel of φ?
   (c) Sketch the image of φ.
   (d) What does the first isomorphism theorem tell you about fifth roots of unity?

4. Suppose an element x of the dihedral group D_n is a composition (in an arbitrary order) of j rotations and k reflections (flips). [Example: $x = r_3 f_2 r_1 r_2 f_1$ with $j = 3$ and $k = 2$] Under what conditions on j and k is x a rotation? A reflection? Explain.

5. Let $\alpha = (1, 2, 5, 4)(2, 6, 3)(5, 6, 3, 2, 1)$ be a permutation (in cycle notation). Express α as a product of disjoint cycles. What is the order of α? Simplify α^{61}.

6. Prove that the set A_n of all even permutations in the symmetric group S_n is a normal subgroup. What can you say about the quotient group S_n/A_n? Give a concrete example of a subgroup of S_3 that is not normal. Explain.

7. How many group homomorphisms from $\mathbf{Z}_{12}$ to $\mathbf{Z}_3 \oplus \mathbf{Z}_4$ are there? How many of them are isomorphisms? If φ is such an isomorphism with $\varphi(2) = [1, 3]$, what is $\varphi(1)$?

8. Suppose R is a commutative ring with unity. Show that the set of all units (elements that have a mutliplicative identity) in R is a multiplicative group under the same multiplication as R.

1	2	3	4	5	6	7	8	total (80)