Final exam / 2015.5.4 / MAT 4233.001 / Modern Abstract Algebra

Name: ____________________

Please show all work and justify your answers.

1. Let $a, b \in \mathbf{N}$. Prove that any common divisor of a and b divides $\gcd(a, b)$.
2. Sketch the subgroup lattice for S_3.
3. Given natural numbers m and n, prove that in $\mathbf{Z}$ we have $\langle m\rangle \cap \langle n\rangle = \langle\, \mathrm{lcm}(m, n)\rangle$.
4. Let τ be the permutation $(1, 3, 2, 5, 6)(2, 3)(4, 6, 5, 1, 2)$. Write τ^{2015} as a product of disjoint cycles.
5. Construct an isomorphism between $\mathrm{Aut}(\mathbf{Z}_m)$ and $U(m)$.
6. Construct an isomorphism between $\mathrm{Aut}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ and S_3.
7. In $U(45)$ find all cosets of the subgroup $\langle 11\rangle$. Explain how your result agrees with predictions of Lagrange's theorem. Determine whether the factor group is cyclic.
8. Let m and n be coprime natural numbers. Suppose G is a commutative multiplicative group with $|G| = m$. Define $\varphi\colon G \to G$ by $\varphi(x) = x^n$. Prove that $\varphi \in \mathrm{Aut}(G)$.
9. Suppose $\varphi : \mathbf{Z}_{33} \to \mathbf{Z}_3 \oplus \mathbf{Z}_{11}$ is a homomorphism and $\varphi(5) = [2, 9]$. Find $\varphi(1)$. Is φ an isomorphism? Explain.
10. Suppose $G < S_n$. Define $\varphi\colon G \to \mathbf{Z}_2$ by $\varphi(\tau) = 0$ if τ is an even permutation and $\varphi(\tau) = 1$ if τ is odd. Prove that φ is a homomorphism. Explain how this proves $A_n \triangleleft S_n$.

1	2	3	4	5	6	7	8	9	10	total (100)