Final / 2009.5.8 / MAT 4233.001 / Modern abstract algebra

Name: ________________________________________

Please show all work and justify your answers.

1. Let $A = \{(x, y) \in \mathbf{R}^2 \colon x + y = 0\}$. Prove that $A$ is an additive subgroup of $\mathbf{R}^2$. Sketch $A$ and two different nontrivial cosets of $A$ in $\mathbf{R}^2$.
2. Suppose $G$ is a group with 81 elements. Prove that $G$ has an element of order 3. Provide an explicit example to show that $G$ need not have an element of order 9.
3. Let $Gl_2(\mathbf{R})$ denote the multiplicative group of invertible $2 \times 2$ matrices with real coefficients and $Sl_2(\mathbf{R})$ denote the subgroup of $Gl_2(\mathbf{R})$ of those matrices with determinant 1. Prove that $Sl_2(\mathbf{R}) \triangleleft Gl_2(\mathbf{R})$ and $Gl_2(\mathbf{R})/Sl_2(\mathbf{R}) \cong \mathbf{R}^*$.
4. Find the isomorphism class of $U(5)$ as a finite abelian group. Explain your reasoning.
5. Prove that a finite integral domain must be a field.
6. Prove or disprove that $\mathbf{Z}_2[x]$ has infinitely many ideals.
7. Let $A = \{p \in \mathbf{Z}_m[x] \colon p(0) = 0\}$. Prove that $A$ is an ideal of $\mathbf{Z}_m[x]$ and $\mathbf{Z}_m[x]/A \cong \mathbf{Z}_m$.
8. Let $A$ be as in the preceding problem. Prove that $A$ is a principal ideal, i.e. can be generated by just one polynomial. For which $m$ is $A$ a prime ideal of $\mathbf{Z}_m[x]$. For which $m$ is $A$ maximal? Explain.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | total (80) |
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