Real Analysis II, MAT 4223
Exam №2, 4/7/93
Instructor: D. Gokhman

Name: ______________________________

1. (40 pts.) For the functions $f\colon [-5,5] \to \mathbf{R}$

   (a) $f(x) = \begin{cases} |x| & \text{for } x \in \mathbf{Q}, \\ -|x| & \text{otherwise.} \end{cases}$ (b) $f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x\cos\left(\frac{1}{x}\right) & \text{otherwise.} \end{cases}$

   (i) Sketch f,

   (ii) For the partition $\{-5, 0, 5\}$ and $\xi_1 = -\pi, \xi_2 = 1$ calculate the Riemann sum $S(P, f, \xi_k)$,

   (iii) For the same partition find $U(P, f)$ and $L(P, f)$ (part (a) only),

   (iv) Find the set of all points of discontinuity of f and determine whether f is Riemann integrable on $[-5, 5]$.

2. (12 pts.) Find **all** monotone increasing α on $[-5, 5]$ such that for any continuous f on $[-5, 5]$
$$\int_{-5}^{5} f\,d\alpha = f(-1) + 3f(1).$$

3. (24 pts.) Suppose $\{f_k\}$ is a sequence of continuous functions which converges uniformly to f and x_k converges to x.

   (a) Prove that $f_k(x_k)$ converges to $f(x)$.

   (b) Find a counterexample to part (a), if we do not require the convergence to be uniform.

4. (24 pts.) Let
$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{(n-1)!}.$$

   (a) Prove that the above series converges uniformly on $\mathbf{R}$.

   (b) Show that $f(x)$ is Riemann integrable on $\left[0, \frac{\pi}{2}\right]$ and
$$\int_0^{\frac{\pi}{2}} f(x)\,dx = e - 1.$$