Midterm 1 / 2016.10.5 / MAT 4213.001 / Real Analysis I

Name: _____

Please show all work. If you use a theorem, name it or state it.

1. Suppose $A = \left\{ \frac{1}{n} : n = 1, 2, ... \right\}$ and $f : A \to \mathbf{R}$ is a bounded function.
   (a) Prove that 0 is a cluster point of A.
   (b) Carefully state the definition of limit and use it to prove that $xf(x) \to 0$ as $x \to 0$.

   Hint: f is bounded means for some m we have $|f(x)| \leq m$ for all $x \in A$.

2. Prove that $\sin x$ fails to have a limit as $x \to \infty$.

3. Find the limit of $\frac{\sin x}{x}$ as $x \to \infty$. Prove your assertion.

4. Consider Dirichlet's function $d(x) = \begin{cases} 1 \text{ for } x \in \mathbf{Q} \\ 0 \text{ otherwise} \end{cases}$
   (a) Prove d is discontinuous at any point.
   (b) Find f discontinuous at every point, but with continuous $|f|$.

5. In each case either find an explicit example of a function $f : \mathbf{R} \to \mathbf{R}$ satisfying the given conditions or state that such a function does not exist. Either way, explain your reasoning.
   (a) f is continuous on the interval $(0, 1)$, but cannot be extended continuously to a larger interval.
   (b) f is continuous on the interval $[0, 1]$, but cannot be extended continuously to a larger interval.

   Hints: A set B is *larger* than A means $A \subseteq B$ and $A \neq B$. If $A \subseteq B$, a continuous function f on A can be *continuously extended* to B means there is a continuous function F on B such that $(\forall x \in A)\ F(x) = f(x)$.

1	2	3	4	5	total (50)