Real Analysis I / MAT 4213.001
Final / December 10, 1998 / Instructor: D. Gokhman

Name: ______________________ Pseudonym: ______________________

Throughout, suppose (x_n) and (y_n) are sequences of real numbers, $x, y \in [-\infty, \infty]$, x is a partial limit of (x_n) and y is a partial limit of (y_n).

1. (20 pts.) Determine whether each of the following statements is true in general. If true, prove it. If false, give a specific counterexample.
   (a) If $\exists m\ \forall n \geq m\ x_n \geq y_n$, then $x \geq y$.
   (b) If $\forall m\ \exists n \geq m\ x_n \geq y_n$, then $x \geq y$.
   (c) If 0 is a partial limit of $x_n - y_n$, then x is a partial limit of (y_n).
   (d) If $x_n - y_n \to 0$, then x is a partial limit of (y_n).

2. (10 pts.) Suppose $A \subseteq \mathbf{R}$ and $a \in \overline{A}$. Prove that there exists a sequence in A that converges to a.

3. (10 pts.) Sketch the following functions and prove that they have no limit at 0:

   (a) $f\colon (0,\infty) \to \mathbf{R}, f(x) = \sin\left(\dfrac{1}{x}\right)$ (b) $f\colon \mathbf{R} \to \mathbf{R}, f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{R} \setminus \mathbf{Q} \end{cases}$

4. (15 pts.) Let f be as in (3b) and let $g(x) = f(x)\sin x$.
   (a) Sketch g.
   (b) Prove that g is continuous at 0.
   (c) Find the set $\{a \in \mathbf{R}\colon g \text{ is continuous at } a\}$.

5. (20 pts.) Suppose $S \subseteq \mathbf{R}$ and $f\colon S \to \mathbf{R}$ is a function. Determine whether each of the following statements is true in general. If true, prove it. If false, give a specific counterexample.
   (a) If $S = \mathbf{R}$, f is continuous, and A is closed in $\mathbf{R}$, then $f^{-1}(A)$ is closed in $\mathbf{R}$.
   (b) If $S = \mathbf{R}$, f is continuous, and A is closed in $\mathbf{R}$, then $f(A)$ is closed in $\mathbf{R}$.
   (c) If $S = \mathbf{R}$, f is continuous, and $\forall x \in \mathbf{Q}\ f(x) = 0$, then $f \equiv 0$.
   (d) If $S = (0,1)$ and f is continuous, then $\exists$ continuous $g\colon [0,1] \to \mathbf{R}$ such that $\forall x \in (0,1)\ g(x) = f(x)$.

6. (10 pts.) Prove that the equation $(x^2+1)^{-2} = x$ has a real solution.

7. (10 pts.) Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is decreasing and $a \in \mathbf{R}$. Prove that $\lim\limits_{x \to a^-} f(x)$ exists.

1	2	3	4	5	6	7	total (95)	%