University of Texas at San Antonio

Real Analysis I, MAT 4213
Final Exam (Extra), FALL 1992, May 4, 1993
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1. (40 pts.)
   (a) Derive the Archimedean law from the Dedekind axiom for the real numbers, i.e. use the fact that any subset of $\mathbf{R}$ which is bounded above has a supremum to show that for any $a, b \in \mathbf{R}, a, b > 0 \; \exists n \in \mathbf{N}$ such that $na > b$. (Hint: Consider the set of all na)
   (b) Suppose (X, d) is a metric space. Let $D \subseteq X$. Prove that D is dense in X, i.e. $\overline{D} = X \Leftrightarrow (\forall \text{ open } U \subseteq X) \; D \cap U \neq \emptyset$.
   (c) Show that any compact metric space is separable, i.e. has a countable dense subset.
   (d) Show that $\mathbf{Q}$ is dense in $\mathbf{R}$, so $\mathbf{R}$ is separable.
   (Hint: Use the Archimedean property and part (b))
2. (20 pts.) Let $\mathcal{L}(E)$ denote the set of all limit points of a set E and $\overline{E}$ denote the closure of E. Show that
   (a) $\mathcal{L}(\mathcal{L}(E)) \subseteq \mathcal{L}(E)$.
   (b) $\mathcal{L}(E) = \mathcal{L}\left(\overline{E}\right)$.
3. (20 pts.) Suppose $E \subseteq K$, where K is compact, and $\mathcal{L}(E) = \emptyset$. Show that E is finite.
4. (20 pts.) Find all cluster points for the following sequences:
   $$\text{(a)} \; \left(1 + \frac{2}{3n}\right)^{4n} \qquad \text{(b)} \; \left(\cos\left(\frac{n\pi}{4}\right)\right)^{((-1)^n)}$$
5. (40 pts.) Determine whether the following series converge.
   $$\text{(a)} \; \sum_{k=1}^{\infty} \frac{3^k + 4^k}{5^k} \qquad \text{(b)} \; \sum_{k=1}^{\infty} \frac{k!}{k^k} \qquad \text{(c)} \; \sum_{k=1}^{\infty} \frac{(-2)^k k^2}{k!} \qquad \text{(d)} \; \sum_{k=1}^{\infty} \sin\left(\frac{\pi}{k}\right)$$
6. (20 pts.) Suppose $f, g : \mathbf{R} \to \mathbf{R}$ are continuous functions. Show that the set $\{x\colon f(x) = g(x)\}$ is closed in $\mathbf{R}$.
7. (20 pts.) Suppose $f : \mathbf{R} \to \mathbf{R}$ satisfies $|f(x)| \leq |x|$ for all x. What is $f(0)$? Prove that f is continuous at $x = 0$.
8. (20 pts.) Classify all functions $f : \mathbf{R} \to \mathbf{R}$ which are continuous and such that $f(\mathbf{R}) \in \mathbf{Q}$.