University of Texas at San Antonio

Real Analysis I, MAT 4213
Final, 12/09/92
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Name: ______________________________

1. (60 pts.) Find examples of series $\sum a_n$, $\sum b_n$ such that
   (a) $\sum a_n$ diverges, $\sum b_n$ diverges, $\sum a_n b_n$ diverges.
   (b) $\sum a_n$ diverges, $\sum b_n$ diverges, $\sum a_n b_n$ converges.
   (c) $\sum a_n$ converges, $\sum b_n$ diverges, $\sum a_n b_n$ diverges.
   (d) $\sum a_n$ converges, $\sum b_n$ diverges, $\sum a_n b_n$ converges.
   (e) $\sum a_n$ converges, $\sum b_n$ converges, $\sum a_n b_n$ diverges.
   (f) $\sum a_n$ converges, $\sum b_n$ converges, $\sum a_n b_n$ converges.

2. (30 pts.) Determine (with proof) the divergence or convergence of the following series:

$$(a)\ \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}} \qquad (b)\ \sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \qquad (c)\ \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

3. (20 pts.) Prove that $\mathbf{R} \setminus \{0\}$ is not connected.

4. (30 pts.) Let $\mathcal{G} = \{(a,b) \colon b - a = 1\}$.
   (a) Prove that $\mathcal{G}$ is a cover of $[0,10]$.
   (b) Exhibit a finite subcover of $[0,10]$ from $\mathcal{G}$.
   (c) What is the smallest possible cardinality of such a subcover?

5. (30 pts.) Suppose $f, g \colon \mathbf{R} \to \mathbf{R}$ are continuous functions.
   (a) Prove that $Z(f) = \{x \in \mathbf{R} \colon f(x) = 0\}$ is closed.
   (b) Suppose $E \subseteq \mathbf{R}$ is dense and $f(x) = g(x)$ for all $x \in E$. Prove that $f(x) = g(x)$ for all $x \in \mathbf{R}$.
   (Hint: use part (a))

6. (30 pts.) Suppose $f : E \to E$ is a continuous function, where $E = [0,1]$.
   (a) Prove that f has a fixed point, i.e. $\exists x \in E\ f(x) = x$.
   (Hint: draw a picture and consider what happens at the end-points)
   (b) Find a counter example for part (a) if $E = (0,1)$.