University of Texas at San Antonio

Real Analysis I, MAT 4213
Final Exam, 12/12/91
Instructor: D. Gokhman

Name: ______________________________

1. (30 pts.) True of false questions, circle your choice. If you choose TRUE, give a brief sketch of proof of the assertion somewhere in the lower part of the page. If FALSE, provide a counterexample.

T F (a) Cartesian product of two countable sets is countable.

T F (b) The set of all positive rational numbers is a Dedekind cut.

T F (c) There is a sequence in $\mathbf{R}$ such that the set of its cluster points is $\mathbf{R}$.

T F (d) Every sequence has a convergent subsequence.

T F (e) If a sequence has exactly one cluster point, then it is convergent.

T F (f) Every open cover of $(0,1)$ has a finite subcover.

2. (25 pts.) Let $\mathcal{S}$ be the collection of all sets and define a relation on $\mathcal{S}$ by $R = \{(A,B)\colon \exists$ a bijection (i.e. a 1-1 correspondence) $f\colon A \to B\}$. Prove that R is an equivalence relation.

3. (25 pts.) Suppose A is a nonempty bounded subset of $\mathbf{Q}$. Let $D = \{r \in \mathbf{Q}\colon \exists\, q \in A$ such that $r > q\}$. Show that D is a Dedekind cut. What is a more familiar name by which we know this real number?

4. (25 pts.) Suppose $f\colon \mathbf{R} \to \mathbf{R}$ satisfies $|f(x)| \le |x|$ for all x. What is $f(0)$? Prove that f is continuous at $x = 0$.

5. (25 pts.) Suppose $f : [0,1] \to [0,1]$ is a continuous function. Show that f has a fixed point, i.e. there exists $x \in [0,1]$ such that $f(x) = x$.

6. (20 pts.) True of false questions, circle your choice. If you choose TRUE, give a brief sketch of proof of the assertion somewhere in the lower part of the page. If FALSE, provide a counterexample.

Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is a continuous function,

T F (a) if $B \subseteq \mathbf{R}$ is a closed set, then $f^{-1}(B)$ is closed.

T F (b) if $B \subseteq \mathbf{R}$ is a compact set, then $f^{-1}(B)$ is compact.

T F (c) if $A \subseteq \mathbf{R}$ is a closed set, then $f(A)$ is closed.

T F (d) if $A \subseteq \mathbf{R}$ is a compact set, then $f(A)$ is compact.

7. (25 pts.) Suppose $f : \mathbf{R} \to \mathbf{R}$ is a differentiable function. Show that if f' is bounded, then f is uniformly continuous.

8. (25 pts.) Let $f\colon [0,1] \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{for } x = \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Sketch f. Construct a partition P of $[0,1]$ such that the lower Riemann sum $L(f,P) \geq \frac{3}{4}$. For the same partition calculate $U(f,P)$.