Final exam / 2015.5.7 / MAT 3233.001 / Modern Algebra

Name: ______________________________

Please show all work and justify your answers.

1. Use the Extended Euclid's Algorithm to find $\gcd(342, 181)$ and $s, t \in \mathbf{Z}$ such that $\gcd(342, 181) = 342s + 181t$. Show steps. What can you conclude about the congruence class of 181 modulo 342?
2. Prove that a common divisor of a and b divides $\gcd(a, b)$.
3. Prove that $3^{\frac{1}{5}}$ is irrational.
4. Suppose $\varphi\colon G \to H$ a homomorphism of multiplicative groups. Use induction to prove that for every natural number n and all $x \in G$ we have $\varphi(x^n) = \varphi(x)^n$.
5. Prove that in the above problem $\varphi(x^{-1}) = \varphi(x)^{-1}$ for all $x \in G$. Use this to extend the result of the above problem to all integers n.
6. Suppose R is a ring, $a \in R$ and $\varphi\colon R \to R$ is an additive group homomorphism defined by $\varphi(x) = ax$. Show that φ is one-to-one if and only if a is neither zero nor a zero divisor.
7. Suppose R in the above problem is a finite commutative ring with unity. Show that the two equivalent statements of the above problem are then also equivalent to a being a unit in R. What does this say about the elements of R?
8. Suppose $\gcd(a, m) = 1$. Given a pair of multiplicative inverses e and d in $U(\varphi(m))$, prove that $(a^e)^d \equiv a \bmod m$.
9. Find all cosets of the subgroup $\langle 11 \rangle < U(50)$. For each coset find its order as an element of the quotient group $U(50)/\langle 11 \rangle$.
10. Find all solutions to the system $5x \equiv 2 \bmod 13$, $2x \equiv 4 \bmod 46$, $x \equiv 3 \bmod 5$.

1	2	3	4	5	6	7	8	9	10	total (100)