Final exam / 2011.12.12 / MAT 3233.001 / Modern Algebra

Name: ______________________________

Please show all work.

1. Prove by induction that $n! \leq n^n$ for all natural numbers n.
2. Suppose a, r, m are natural numbers with $a \equiv r \bmod m$. Prove that $\gcd(a, m) = \gcd(r, m)$.
3. Let m, n be positive integers, $m > 1$. Prove that the following are equivalent.
   (i) $\gcd(m, n) = 1$
   (ii) the congruence class $[n]_m$ is a unit in the ring $\mathbf{Z}_m$
   (iii) $[n]_m$ generates the additive group $\mathbf{Z}_m$
4. Let $H = \{(), (1,2)(3,4), (1,3)(2,4), (2,3)(1,4)\} \subset S_4$ (symmetric group).
   (a) Show that H is a subgroup of S_4.
   (b) Is H cyclic? Explain.
   (c) Partition S_4 with left cosets of H.
5. Suppose $\varphi \colon G \to G'$ is a homomorphism of multiplicative groups.
   (a) Prove that $\ker \varphi$ is a subgroup of G.
   (b) Prove that $\varphi(G)$ is a subgroup of G'.
   (c) Suppose $\varphi(x) = y$. Prove that the fibre $\varphi^{-1}(y)$ is the coset $x \ker \varphi$.
6. Find the solution set for the system of congruences
$$15x \equiv 10 \bmod 50$$
$$x \equiv 9 \bmod 15$$
7. Use Euclid's algorithm for the polynomial ring $\mathbf{R}[x]$ to find the greatest common divisor and the Bézout coefficients for $x^2 - x - 6$ and $x^4 + 2x^3 - x - 2$.
8. Suppose $\varphi \colon \mathbf{R}[x] \to \mathbf{R}$ is the evaluation map $\varphi(p(x)) = p(0)$.
   (a) Prove that φ is a ring homomorphism.
   (b) Prove that φ is onto.
   (c) Prove that $\ker \varphi$ is the set of all polynomials with zero constant coefficient.

1	2	3	4	5	6	7	8	total (80)	%