Final Exam / 2012.5.7 / MAT 3013.002 / Foundations of Mathematics

Name: ____________________

Please show all work.

1. Prove by induction that $4^n > n^4$ for all natural numbers $n \geq 5$.
2. Determine whether each of the statements is a tautology, a contradiction or neither.

   (a) $((p \to q) \to p) \to q$ (b) $((p \vee q) \to q) \to p$

3. Negate the statement $(\forall x)(\exists y)(\forall z)[p(x, y) \leftrightarrow q(y, z)]$ and simplify.
4. Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is twice continuously differentiable function. Prove that the following collections are sets. You may assume $\mathbf{R}$ is a set.

   (a) zeros of f (b) critical points of f (c) inflection points of f

5. Prove that $(A \setminus B) \times (C \setminus D) \subseteq (A \times C) \setminus (B \times D)$. Provide a concrete counter example illustrating why containment the other way fails.
6. Suppose $f\colon X \to Y$ is a function and $A, B \subseteq X$. Prove that $f_*(A \cap B) \subseteq f_*(A) \cap f_*(B)$. Provide a concrete counter example illustrating why containment the other way fails.
7. Suppose $f\colon X \to Y$ is a function and $C, D \subseteq Y$. Prove that $f^*(C \cap D) = f^*(C) \cap f^*(D)$.
8. Let $\pi_1\colon \mathbf{R}^2 \to \mathbf{R}$ be the natural projection to the first coordinate, i.e. $\pi_1([x, y]) = x$. Define two points in $\mathbf{R}^2$ to be equivalent when their values under π_1 are the same. Prove that this is an equivalence relation. Describe geometrically the equivalence classes. Sketch.
9. Prove that if $g \circ f$ is onto, then g is onto. Provide a concrete counterexample of how f need not be onto, even if $g \circ f$ is the identity function — make sure you specify the domains and co-domains.
10. Suppose $S = \{x \in \mathbf{Q}\colon x^3 < 0\}$. If they exist, what are the least and greatest elements of S. For S as a subset of $\mathbf{R}$, same question for $\sup S$ and $\inf S$. Prove your assertions.

1	2	3	4	5	6	7	8	9	10	total (100)