Final exam / 2016.12.12 / MAT 2233.002 / Linear Algebra

Name: ____

Please show all work. Check your answers! ◡̈

1. Consider the linear system $\begin{bmatrix} 2x+4y+6z=0 \\ 3x+4y+5z=4 \\ 6x+7y+9z=0 \end{bmatrix}$. Use Gauss-Jordan elimination to find all solutions. Show steps. Describe and sketch the solution set. Can you expect some solutions to this system for arbitrary right-hand-sides? Explain.

2. Suppose $T\colon \mathbf{R}^2 \to \mathbf{R}$ is a linear map and we know its values at some two (column) vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^2$ that not scalar multiples of one another: $T(\mathbf{u}) = a, T(\mathbf{v}) = b$.

   (a) Let $S = [\mathbf{u}, \mathbf{v}]$. Explain why S is an invertible matrix. What is $\text{rref}(S)$?

   (b) Let A be the matrix that represents T. Let $B = [a, b]$. Explain why $AS = B$.

   Hint: Since $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^2$, $A = [A\mathbf{e}_1, A\mathbf{e}_2] = [T(\mathbf{e}_1), T(\mathbf{e}_2)]$, so compute $AS\mathbf{e}_i = T(S\mathbf{e}_i) = ...$

3. Preceding problem continued:

   (c) Use (a) to solve the matrix equation in (b) for A.

   (d) If $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, $a = 1$ and $b = -2$ use your solution in (c) to find A.

4. In each part enter a real 2×2 nonzero nonidentity matrix A such that the linear map $\mathbf{x} \mapsto A\mathbf{x}$ is as given.

   $\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$ (a) orthogonal projection to the main diagonal

   $\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$ (b) orthogonal reflection with respect to the main diagonal

   $\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$ (c) isotropic dilation

   $\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$ (d) rotation by $\frac{\pi}{2}$

   $\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$ (e) vertical shear

CONTINUED ON THE OTHER SIDE

1	2	3	4	5	6	7	8	total (80)

5. Suppose V is an inner product space and U is a subspace of V. Prove that $U^\perp$ is also a subspace of V. Show that any vector in V can be expressed uniquely as a sum of two vectors, one in U and the other in $U^\perp$ (this is the main idea behind Gram-Schmidt).

6. Let P_2 be the vector space of all real polynomials $p(t)$ with degree ≤ 2 and let $\varepsilon\colon P_2 \to \mathbf{R}$ be the evaluation map: $\varepsilon(p(t)) = p(0)$.

   (a) Prove that ε is linear. What is the rank of ε? What is the dimension of $\ker \varepsilon$?

   (b) Describe $\ker \varepsilon$ and find an orthonormal basis for it relative to the inner product $\langle p(t), q(t) \rangle = \displaystyle\int_0^1 p(t)q(t)\,dt$.

7. Let $A = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 6 & 7 & 2 & 0 \\ 3 & 0 & 0 & 4 \\ 3 & 2 & 1 & 0 \end{bmatrix}$ and define $T\colon \mathbf{R}^4 \to \mathbf{R}^4$ by $T(\mathbf{x}) = A\mathbf{x}$. Compute the determinant of A (show work). What can you conclude about T from your answer?

8. Let $A = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}$.

   (a) Find the eigenvalues of A and corresponding eigenvectors. Let S be the matrix whose columns are eigenvectors of A. Compute AS. Verify that $S^{-1}AS$ is diagonal with entries the eigenvalues of A.

   (b) Sketch the eigenspaces and give a geometrical description of the linear map $\mathbf{x} \mapsto A\mathbf{x}$.