

Name: _____

Please show all work. Check your answers! 😊

1. Consider the linear system $\begin{cases} 2x + 4y + 6z = 0 \\ 3x + 4y + 5z = 4 \\ 6x + 7y + 9z = 0 \end{cases}$. Use Gauss-Jordan elimination to find all solutions. Show steps. Describe and sketch the solution set. Can you expect some solutions to this system for arbitrary right-hand-sides? Explain.

2. Suppose $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is a linear map and we know its values at some two (column) vectors \mathbf{u} and \mathbf{v} in \mathbf{R}^2 that not scalar multiples of one another: $T(\mathbf{u}) = a, T(\mathbf{v}) = b$.

(a) Let $S = [\mathbf{u}, \mathbf{v}]$. Explain why S is an invertible matrix. What is $\text{rref}(S)$?

(b) Let A be the matrix that represents T . Let $B = [a, b]$. Explain why $AS = B$.

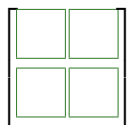
Hint: Since $T(\mathbf{x}) = A\mathbf{x}$ for all \mathbf{x} in \mathbf{R}^2 , $A = [A\mathbf{e}_1, A\mathbf{e}_2] = [T(\mathbf{e}_1), T(\mathbf{e}_2)]$, so compute $AS\mathbf{e}_i = T(S\mathbf{e}_i) = \dots$

3. Preceding problem continued:

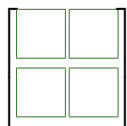
(c) Use (a) to solve the matrix equation in (b) for A .

(d) If $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, $a = 1$ and $b = -2$ use your solution in (c) to find A .

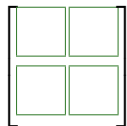
4. In each part enter a real 2×2 nonzero nonidentity matrix A such that the linear map $\mathbf{x} \mapsto A\mathbf{x}$ is as given.



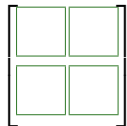
(a) orthogonal projection to the main diagonal



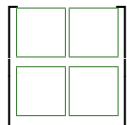
(b) orthogonal reflection with respect to the main diagonal



(c) isotropic dilation



(d) rotation by $\frac{\pi}{2}$



(e) vertical shear

CONTINUED ON THE OTHER SIDE

1	2	3	4	5	6	7	8	total (80)

5. Suppose V is an inner product space and U is a subspace of V . Prove that U^\perp is also a subspace of V . Show that any vector in V can be expressed uniquely as a sum of two vectors, one in U and the other in U^\perp (this is the main idea behind Gram-Schmidt).
6. Let P_2 be the vector space of all real polynomials $p(t)$ with degree ≤ 2 and let $\varepsilon: P_2 \rightarrow \mathbf{R}$ be the evaluation map: $\varepsilon(p(t)) = p(0)$.
- Prove that ε is linear. What is the rank of ε ? What is the dimension of $\ker \varepsilon$?
 - Describe $\ker \varepsilon$ and find an orthonormal basis for it relative to the inner product

$$\langle p(t), q(t) \rangle = \int_0^1 p(t)q(t) dt.$$
7. Let $A = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 6 & 7 & 2 & 0 \\ 3 & 0 & 0 & 4 \\ 3 & 2 & 1 & 0 \end{bmatrix}$ and define $T: \mathbf{R}^4 \rightarrow \mathbf{R}^4$ by $T(\mathbf{x}) = A\mathbf{x}$. Compute the determinant of A (show work). What can you conclude about T from your answer?
8. Let $A = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}$.
- Find the eigenvalues of A and corresponding eigenvectors. Let S be the matrix whose columns are eigenvectors of A . Compute AS . Verify that $S^{-1}AS$ is diagonal with entries the eigenvalues of A .
 - Sketch the eigenspaces and give a geometrical description of the linear map $\mathbf{x} \mapsto A\mathbf{x}$.