Final exam / 2016.5.5 / MAT 2233.002 / Linear Algebra

Name: ______________________________

Please show all work.

1. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$. Find all solutions to $Ax = b$. Sketch/describe the solution set. Compute $\det(A)$. Can you expect some solutions to $Ax = b$ for any b? Explain.

2. Let $A = \begin{bmatrix} 1 & 2 & 5 & 6 \\ 1 & 1 & 7 & 8 \end{bmatrix}$. Find a basis for the kernel of A. What is the image of A?

3. Find the matrix A relative to the standard basis for rotation by -90 degrees. Same for reflection with respect to the main diagonal. In each case compute A^4 and explain your result geometrically.

4. Let P_2 be the space of all real polynomials $p(t)$ with degree ≤ 2 and let $T\colon P_2 \to P_2$ be the linear map given by $T(p) = p(t+1) - p(t-1)$. Find the matrix that represents T with respect to the basis $[1, t, t^2]$.

5. Let $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. What matrix represents the linear map $\mathbf{x} \mapsto A\mathbf{x}$ relative to the basis $[\mathbf{v}_1, \mathbf{v}_2]$? Express $A\mathbf{v}_1$ and $A\mathbf{v}_2$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.

6. Prove that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the plane $x + y + z = 0$. Let M be the matrix with columns $\mathbf{v}_1$ and $\mathbf{v}_2$. Find the QR factorization of M.

7. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix}$. Find the least squares solution x^* to $Ax = b$ and verify that $b - Ax^*$ is orthogonal to the image of A.

8. (a) Let $A = \begin{bmatrix} -9 & 8 \\ -12 & 11 \end{bmatrix}$. Find the eigenvalues of A and corresponding eigenvectors.
   (b) Let S be the matrix whose columns are eigenvectors of A. Verify that $S^{-1}AS$ is diagonal with entries the eigenvalues of A.
   (c) Sketch the eigenspaces and give a geometrical description of the linear map $\mathbf{x} \mapsto A\mathbf{x}$.

9. Let A be an upper-triangular matrix (all entries below the diagonal are zero). Show that $\det(A)$ is the product of its diagonal entries. Then show that the diagonal entries are eigenvalues of A.

1	2	3	4	5	6	7	8	9	total (90)