ADVANCED EXAMINATION COMPLEX ANALYSIS Summer 2004

Name: ______________________________

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Choose 8 questions to answer and indicate your choices in the top parts of the boxes above. Please supply brief narration with your formulas, state the results you use, and show all work!

1. Evaluate the following integrals around the unit circle: (a) $\int \frac{dz}{\cos z}$ (b) $\int \frac{dz}{z^2 + 2z}$
2. Find the Laurent series for $f(z) = \frac{1}{iz-1}$ valid for: (a) $|z| < 1$ (b) $|z| > 1$.
3. Construct a Möbius transformation taking the main diagonal $y = x$ to the unit circle. Prove that the transformation you constructed does what is claimed. What can you say about uniqueness of such a transformation? State the various properties of Möbius transformations that you use in your proof.
4. Suppose $f$ is analytic in a punctured open disc $D^*$ centered at the origin. What is the relationship between the integral of $f(z)\,dz$ around a circle in $D^*$ centered at 0 (called the residue of $f$ at 0) and the Laurent expansion of $f$ at 0? Prove your assertion.
5. Suppose $g$ is holomorphic in a neighborhood of the origin and $g(0) = g'(0) = 0$.

   Show that $h(z) = z^{-2}g(z)$ has a removable singularity at 0.
6. State and prove the Riemann extension theorem on removable singularities.
7. State and prove the theorem of Weierstrass on convergent sequences of analytic functions and their derivatives.
8. Determine the number of zeros of $f(z) = z^5 + 3iz - 1$ in the annulus $\{z : 1 < |z| < 2\}$. Justify your assertion. You may use Rouché's theorem.
9. Derive the formula for Taylor coefficients of a holomorphic function using the harmonic series and Cauchy's integral formula. You may use general results about uniform convergence.
10. Let $f(z) = (e^z - 1)^{-1}z$ if $z \neq 0$, and $f(0) = 1$.
    (a) Find all singularities of $f$.
    (b) Find the first two nontrivial terms of the Taylor series for $f$ at $z = 0$.
11. Prove the uniqueness of a Möbius transformation with prescribed values at 3 distinct points.
12. State and prove Schwartz's lemma.