ADVANCED EXAMINATION · COMPLEX ANALYSIS · Summer 2003

Name: ____________________

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Choose 8 questions to answer. Enter the selected problems in the top parts of the boxes above. Please supply brief narration with your formulas, state the results you use, and show all work!

1. Suppose f is holomorphic on a domain. Prove that f is constant
   (a) if the real part $\mathcal{R}[f]$ is constant.
   (b) if the modulus $|f|$ is constant.

2. Evaluate the following integrals around the unit circle: (a) $\displaystyle\int \frac{dz}{\sin z}$ (b) $\displaystyle\int \frac{dz}{z^3 - 2iz^2}$

3. Expand $f(z) = \dfrac{1}{z^2 + 3iz - 2}$ in a Laurent series valid in the annulus $\{z \in \mathbf{C}\colon 1 < |z| < 2\}$.

4. Construct a Möbius transformation taking the real axis to the unit circle. Prove that the transformation you constructed does what is claimed. You may use various properties of Möbius transformations in your proof.

5. Suppose $f \not\equiv 0$ has the Laurent expansion at the origin $\displaystyle\sum_{k=n}^{\infty} a_k z^k$, where $n \in \mathbf{Z}$. Prove that
   (a) $\exists$ punctured disc D^* around 0 containing neither singularities nor zeros of f.
   (b) $2\pi i\, a_{-1}$ is the integral of f along a circle in the interior of D^* around 0.

6. Suppose f is entire. Prove that $M(r) = \max\limits_{|z|=r} |f(z)|$ is a nondecreasing function of r.

7. Let $r > 0$. Derive Cauchy's inequalities for a function f analytic at z_0

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|$$

8. Let $M, p > 0$. Suppose $f(z)$ is entire and $|f(z)| \leq M\,|z|^p$ for all z with $|z|$ sufficiently large. Prove that f is a polynomial with $\deg f \leq p$. You may use Cauchy's inequalities.

9. Let D denote the unit disc. Suppose $g\colon D \to D$ is holomorphic with $g(0) = 0$.
   (a) Show that $h(z) = g(z)/z$ has a removable singularity at 0.
   (b) Prove that $|g(z)| \leq |z|$ and $|g'(0)| \leq 1$.

10. State and prove the Riemann extension theorem on removable singularities.

11. State and prove the principle of analytic continuation.

12. State and prove the theorem of Weierstrass on convergent sequences of analytic functions.