ADVANCED EXAMINATION □ COMPLEX ANALYSIS □ April 29, 1997

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Name: ______________________________

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Work any eight problems. Indicate which problems you are doing in the top parts of the boxes above.

Throughout, unless otherwise indicated, assume that Ω is a lattice in $\mathbf{C}$; D is a domain in $\mathbf{C}$; and Σ is the Riemann sphere.

1. State and prove the Maximum Modulus Principle. You may use one of the following (a) Cauchy's Integral Formula, (b) the Open Mapping Theorem.

2. State and prove Liouville's theorem. You may use Cauchy's Integral Formula.

3. Suppose $f(z)$ is entire and $|f(z)| \leq |z|$ for all z. Prove that $f(z)$ is linear.

4. (a) Evaluate $\displaystyle\int \frac{dz}{\tan z}$ around the unit circle.

   (b) Evaluate $\displaystyle\int_0^{2\pi} e^{e^{i\theta}}\, d\theta$. (Hint: Let $z = e^{i\theta}$.)

5. Find the Laurent series for $\dfrac{1}{z^2 - 3iz - 2}$ valid in the annulus $\{z \in \mathbf{C}\colon 1 < |z| < 2\}$.

6. Suppose $f(z) = \displaystyle\sum_{k=-\infty}^{\infty} a_k z^k$ near 0. Prove that the residue of $f(z)$ at 0 is a_{-1}.

7. Suppose $f\colon \mathbf{C} \to \mathbf{C}$ is entire and $|f|$ is constant. Prove that $f(z)$ is constant.

8. Suppose w_1, w_2, w_3 are distinct points of $\mathbf{C}$. Prove that there exists a unique Möbius transformation $T\colon \Sigma \to \Sigma$ such that $T(0) = w_1$, $T(1) = w_2$, and $T(\infty) = w_3$.

9. Suppose $f\colon \Sigma \to \Sigma$ is meromorphic. Prove that

   (a) The number of poles of f is finite.

   (b) f is a rational function.

10. Suppose $f\colon \Sigma \to \Sigma$ is a rational function and $g\colon \mathbf{C}/\Omega \to \Sigma$ is an elliptic function. For each of f and g prove the following

    (a) The number of poles is finite.

    (b) The number of poles equals the number of zeros (counted with multiplicity).

11. Let $F_k(z) = \displaystyle\sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{(z-\omega)^k}$ with $k \geq 3$.

    (a) Show that the above series for $F_k(z)$ converges normally on $\mathbf{C} \setminus \Omega$.

    (b) What are the poles of $F_k(z)$ and what is their multiplicity?

    (c) Prove that $F_k(z)$ is elliptic.

12. Let $\mathcal{P}(z) = \dfrac{1}{z^2} + \displaystyle\sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$. Suppose $\mathcal{P}(z_1) = \mathcal{P}(z_2)$. Prove that $z_1 \pm z_2 \in \Omega$.