ADVANCED EXAMINATION □ COMPLEX ANALYSIS □ October 13, 1995

Dr. Dmitry Gokhman / Division of Mathematics and Statistics / UT San Antonio

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Work any eight problems. Indicate which problems you are doing in the top parts of the boxes above.

Throughout, unless otherwise indicated, assume that Ω is a domain, i.e. an open connected subset of the complex plane $\mathbf{C}$.

1. Suppose $f(z)$ is continuous on $\mathbf{C}$ and holomorphic on $\Omega = \mathbf{C} \setminus \{0\}$. Prove that f is entire (i.e. holomorphic on $\mathbf{C}$).

2. (a) Derive the formula for Taylor coefficients of a holomorphic function using the harmonic series $1/(1-z) = \sum_{k=0}^{\infty} z^k$ and Cauchy's integral formula. You may use general results about uniform convergence.

   (b) Let $r > 0$. Derive Cauchy's Inequalities:

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|$$

3. Let $M, p > 0$. Suppose $f(z)$ is entire and $|f(z)| \leq M\,|z|^p$ for all z with $|z|$ sufficiently large. Use Cauchy's inequalities to prove that f is a polynomial with $\deg f \leq p$.

4. (a) By computing $d(f(z)\,dz)$ show that $f(z)$ is holomorphic on Ω if, and only if, $f(z)\,dz$ is closed.

   (b) Show that dz/z is closed on $\Omega = \mathbf{C} \setminus \{0\}$. Is dz/z exact on Ω? Is Ω simply connected?

5. Prove the Fundamental Theorem of Algebra: Every nonconstant polynomial with complex coefficients has a complex root. You may use one of the following (a) Liouville's theorem, (b) the Maximum Modulus Principle, (c) Rouché's theorem.

6. (a) Suppose $f(z)$ holomorphic and nonconstant on Ω. Apply the Maximum Modulus Principle to $1/f$ to show that if $\min f(z)$ is attained at $z_0 \in \Omega$, then $f(z_0) = 0$.

   (b) Suppose $f(z)$ is holomorphic on Ω and continuous on the closure $\overline{\Omega}$. Further assume that $|f(z)|$ is nonconstant on Ω and constant on the boundary $\partial\Omega$. Prove that f has a zero in Ω.

7. Suppose $f(z)$ is entire. Prove that $M(r) = \max_{|z|=r} |f(z)|$ is a nondecreasing function of r.

8. Suppose $f(z)$ is entire and $|f(z)| \leq e^{\operatorname{Re} z}$ for all z. What can you say about f?

9. Suppose $z_0 \in \Omega$ and $h(z)$ is holomorphic on Ω.

   (a) Use the Taylor expansion of $h(z)$ at z_0 to prove that if $h \not\equiv 0$ and $h(z_0) = 0$, then there exists $k > 0$ such that $h(z) = (z - z_0)^k g(z)$, where $g(z_0) \neq 0$.

   (b) Prove that if $h \not\equiv 0$, then the zeros of h are isolated.

   (c) Suppose that $D \subseteq \Omega$ is nonempty and open. Prove that if two holomorphic functions h_1 and h_2 agree on D, then $h_1 \equiv h_2$ on Ω.

10. Suppose $f(z)$ is holomorphic on Ω. Prove that if either $\operatorname{Re} f$, $\operatorname{Im} f$, or $|f|$ are constant on Ω, then f is constant on Ω.

11. Suppose $G = \{z \in \mathbf{C}\colon\ |z| < 1\}$ and $g\colon G \to G$ is holomorphic with $g(0) = 0$.

    (a) Show that $h(z) = g(z)/z$ has a removable singularity at 0. What should $h(0)$ be to make h holomorphic on G?

    (b) Apply the Maximum Modulus Prinicple to $h(z)$ to show that $|g(z)| \leq |z|$ and $|g'(0)| \leq 1$.

12. Use Rouché's theorem to find the number of zeros of $f(z) = z^7 - 5z^4 + z^2 - 2$ inside the unit circle.

13. (a) Evaluate the following integral around the unit circle

$$\int \frac{dz}{\sin z}$$

    (b) Use Cauchy integration with $z = e^{\mathbf{i}\theta}$ to evaluate the integral

$$\int_0^{2\pi} e^{e^{\mathbf{i}\theta}}\, d\theta.$$

14. Let $f(z) = 1/(z + i)$.

    (a) Find the Laurent series for f valid for $|z| < 1$

    (b) Find the Laurent series for f valid for $|z| > 1$