# ADVANCED EXAMINATION / ALGEBRA / Summer, 2012

THE UNIVERSITY OF TEXAS AT SAN ANTONIO

Name: ______________________________

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Work any 8 problems. Indicate which problems you are doing in the top parts of the boxes above. Please show all work and either name or state the results you use in your proofs.

Notation: $K$: commutative ring with unity, $D$: integral domain, $F$: field, $n$: positive integer.

1. Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ is a short exact sequence of $K$-modules and $C$ is free. Prove that $B \cong A \oplus C$.
2. How many group endomorphisms does $\mathbf{Z}_{12}$ have? Exhibit all group automorphisms of $\mathbf{Z}_{12}$. What famous group is $\mathrm{Aut}(\mathbf{Z}_{12})$ isomorphic to? Explain.
3. Suppose $f\colon A \to B$ is a $K$-module morphism. State and prove the universal property of the canonical injection $i\colon \ker f \to A$.
4. Prove that the set of units $U(K)$ is a multiplicative group. Prove that $K$ is a local ring if and only if $K \setminus U(K)$ is a maximal ideal.
5. Prove that (a) $\mathbf{Z}_5 \otimes_{\mathbf{Z}} \mathbf{Z}_7 \cong 0$, and (b) $\mathbf{Z}^n \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbf{R}^n$.
6. Let $\mathcal{S}$ be the set of all subgroups of the symmetric group $S_3$. Define $f : S_3 \times \mathcal{S} \to \mathcal{S}$ by $f(\sigma, S) = \sigma S \sigma^{-1}$.
   (a) Prove that $f$ is a group action.
   (b) Compute the orbit of the subgroup $H$ generated by the 3-cycle $(1, 2, 3)$ and the orbit of the alternating group $A_3$.
   (c) What are the normalizers $N_{S_3}(H)$ and $N_{S_3}(A_3)$? Prove your assertions.
7. Let $A$ be a $K$-module and define $f\colon A \times K^n \to A^n$ by $f(a, [\kappa_n]) = [\kappa_n a]$.
   (a) Prove that $f$ is $n$-linear.
   (b) Prove that $f$ is universal among $n$-linear maps on $A \times K^n$.
8. Prove that the symmetric group $S_3$ is solvable, but not nilpotent.
9. State and prove the Nakayama Lemma.
10. Prove that the free abelian group functor from sets to abelian groups and the forgetful functor from abelian groups to sets are adjoints.
11. Prove that the Krull dimension of the polynomial ring $F[x_1, ...x_n]$ is $n$.