ADVANCED EXAMINATION / ALGEBRA / February 11, 2005

THE UNIVERSITY OF TEXAS AT SAN ANTONIO

Name: ____

#	#	#	#	#	#	#	#	total (80)

Work any 8 problems. Indicate which problems you are doing in the top parts of the boxes above. Please show all work and either name or state the results you use in your proofs.

Unless otherwise indicated, R denotes a ring, K a commutative ring, and D an integral domain.

1. Suppose $f\colon R \to K$ is a morphism of commutative rings, J is an ideal of R and I is an ideal of K. Prove or disprove: (a) $f^{-1}(I)$ is an ideal of R, (b) $f(J)$ is an ideal of K.

2. Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ is a short exact sequence of K-modules and C is free.
   Prove that $B \cong A \oplus C$.

3. (a) Prove that the set of units of K forms a multiplicative group.
   (b) What are the multiplicative groups of units of the rings $\mathbf{Z}^2$ and $\mathbf{Z}_4$?

4. Suppose $f\colon A \to B$ is a K-module morphism. State and prove the universal property of the projection $p\colon B \to \operatorname{coker} f$.

5. Prove that the dual of a K-module monomorphism is an epimorphism.

6. Suppose F is free K-module on 2 generators. Prove from first principles that the set of all bilinear alternating maps $F^2 \to K$ is a free cyclic K-module.

7. Prove or disprove: (a) maximal ideals of K are prime, (b) prime ideals of K are maximal.

8. Compute (a) $\mathbf{Z}_3 \otimes_{\mathbf{Z}} \mathbf{Z}^2$, (b) $\mathbf{Z}_2 \otimes_{\mathbf{Z}} \mathbf{Q}$. Prove your assertions.

9. Let $\mathcal{S}$ be the set of all subgroups of the symmetric group S_3. Define $f : S_3 \times \mathcal{S} \to \mathcal{S}$ by $f(\sigma, S) = \sigma S \sigma^{-1}$.
   (a) Prove that f is a group action.
   (b) Compute the orbit of the subgroup H generated by the involution $(1, 2)$ and the orbit of the alternating group A_3.
   (c) What are the normalizers $N_{S_3}(H)$ and $N_{S_3}(A_3)$? Prove your assertions.

10. Let A be a K-module and define $f\colon A \times K^2 \to A^2$ by $f(a, (\kappa, \lambda)) = (\kappa a, \lambda a)$.
    (a) Prove that f is bilinear.
    (b) Prove that f is universal among bilinear maps on $A \times K^2$ by showing that for any bilinear $g\colon A \times K^2 \to C$ there exists unique linear $g'\colon A^2 \to C$ with $g = g' \circ f$.

11. Prove that the commutator subgroup of the dihedral group Δ_n is cyclic.