ADVANCED EXAMINATION / ALGEBRA / November 5, 2004

THE UNIVERSITY OF TEXAS AT SAN ANTONIO

Name: ______________________________

#	#	#	#	#	#	#	#	total (80)

Work any 8 problems. Indicate which problems you are doing in the top parts of the boxes above. Please show all work and either name or state the results you use in your proofs.

Unless otherwise indicated, R denotes a ring, K a commutative ring, and D an integral domain.

1. Prove that any maximal ideal in a commutative ring is a prime ideal.

2. Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ is a short exact sequence of K-modules and C is free. Prove that $B \cong A \oplus C$.

3. How many group endomorphisms does $\mathbf{Z}_{12}$ have? Exhibit all group automorphisms of $\mathbf{Z}_{12}$. What famous group is $\mathrm{Aut}(\mathbf{Z}_{12})$ isomorphic to? Explain.

4. Suppose $f\colon A \to B$ is a K-module morphism. State and prove the universal property of the projection $p\colon B \to \operatorname{coker} f$.

5. Prove that the dual of an K-module epimorphism is a monomorphism.

6. (a) Show that if D has exactly one maximal ideal M, then $D \setminus M$ is the multiplicative group of units of R.
   (b) Show that D is a field $\Leftrightarrow$ D has no proper nonzero ideals.

7. Prove that (a) $\mathbf{Z}_2 \otimes_{\mathbf{Z}} \mathbf{Z}_3 \cong 0$, and (b) $\mathbf{Z}^2 \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}^2$.

8. Let $\mathcal{S}$ be the set of all subgroups of the symmetric group S_3. Define $f : S_3 \times \mathcal{S} \to \mathcal{S}$ by $f(\sigma, S) = \sigma S \sigma^{-1}$.

   (a) Prove that f is a group action.
   (b) Compute the orbit of the subgroup H generated by the involution $(1, 2)$ and the orbit of the alternating group A_3.
   (c) What are the normalizers $N_{S_3}(H)$ and $N_{S_3}(A_3)$? Prove your assertions.

9. Let A be a K-module and define $f\colon A \times K^2 \to A^2$ by $f(a, (\kappa, \lambda)) = (\kappa a, \lambda a)$.
   (a) Prove that f is bilinear.
   (b) Prove that f is universal among bilinear maps on $A \times K^2$ by showing that for any bilinear $g\colon A \times K^2 \to C$ there exists unique linear $g'\colon A^2 \to C$ with $g = g' \circ f$.

10. Prove that the symmetric group S_3 is solvable, but not nilpotent.

11. Prove that every finite integral domain is a field.