

Velocity field generated by singularities in a thin lamina of variable thickness

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The present note considers the velocity field in a thin lamina of variable thickness generated by singularities as applied to the problem of flow past a cascade of blades of a radial-axial hydroturbine.

I. In the x, y plane consider a thin lamina of thickness $h = h(y)$ satisfying the following conditions: $d \ln h / dy$ is continuous and does not change sign, $\lim_{y \rightarrow \infty} h = h_{(\infty)}$ and $\lim_{y \rightarrow -\infty} h = h_{(-\infty)}$, where $0 < h_{(\infty)} < h_{(-\infty)} < \infty$. Let us place an infinite sequence of sources $Qh_{(0)}$ along the x -axis with period t ($h_{(0)}$ is the thickness of the lamina at $y = 0$). If we neglect the change in all quantities with respect to z at each point (x, y) and assume that $\partial \Phi_s / \partial z = 0$, where Φ_s is the velocity potential, then the differential equation for the potential due to such a sequence of sources is [1]

$$\frac{\partial^2 \Phi_s}{\partial x^2} + \frac{d \ln h}{dy} \frac{\partial \Phi_s}{\partial y} + \frac{\partial^2 \Phi_s}{\partial y^2} = Q \delta(y) \sum_{n=-\infty}^{\infty} \delta(x - nt), \quad (1)$$

where $\delta(y)$ and $\delta(x - nt)$ are Dirac delta-functions.

In [1] it is proposed to find the solution of equation (1) in the form of a Fourier series, whose coefficients are determined from a system of ordinary differential equations with boundary conditions

$$\left[h \frac{\partial \Phi_s}{\partial y} \right]_{y=\infty} = - \left[h \frac{\partial \Phi_s}{\partial y} \right]_{y=-\infty} = \frac{1}{2t} Q h_{(0)}. \quad (2)$$

Let us show that besides the flow from the sequence of sources, the solution obtained for boundary conditions (2) will have a uniform flow

$$v_u = Q h_{(0)} \frac{h_{(-\infty)} - h_{(\infty)}}{2ht[h_{(-\infty)} + h_{(\infty)}]},$$

i. e. the boundary conditions for solving equation (1) are

$$\left[\frac{\partial\Phi_s}{\partial y}\right]_{y=\infty} = -\left[\frac{\partial\Phi_s}{\partial y}\right]_{y=-\infty} = \frac{Qh_{(0)}}{t[h_{(-\infty)} + h_{(\infty)}]}, \quad (3)$$

II. Let us write the equation for the potential due a concentrated source at the origin $(0, 0)$:

$$\frac{\partial^2\Phi_o}{\partial x^2} + \frac{d \ln h}{dy} \frac{\partial\Phi_o}{\partial y} + \frac{\partial^2\Phi_o}{\partial y^2} = Q\delta(x)\delta(y). \quad (4)$$

We can decompose $\Phi_o = \Phi_{o_0} + \phi'_o + \phi''_o$, into Φ_{o_0} — potential of flow in a lamina of constant thickness and $\phi_o = \phi'_o + \phi''_o$ — potential of additional flow caused by the variability of thickness of the lamina. We have

$$\begin{aligned} \frac{\partial^2\Phi_{o_0}}{\partial x^2} + \frac{\partial^2\Phi_{o_0}}{\partial y^2} &= Q\delta(x)\delta(y), \\ \frac{\partial^2\phi'_o}{\partial x^2} + \frac{\partial^2\phi'_o}{\partial y^2} &= -\frac{d \ln h}{dy} \frac{\partial\Phi_{o_0}}{\partial y}, \\ \frac{\partial^2\phi''_o}{\partial x^2} + \frac{\partial^2\phi''_o}{\partial y^2} &= -\frac{d \ln h}{dy} \frac{\partial\phi_o}{\partial y}; \end{aligned} \quad (5)$$

with boundary conditions $\nabla\phi'_o|_{r=\infty} = 0; \nabla\phi''_o|_{r=\infty} = 0$, where $r = \sqrt{x^2 + y^2}$, because $h_{(\pm\infty)} \neq 0$.

With the aid of a Green's function we find the solution with ϕ'_o satisfying the imposed boundary conditions:

$$\phi'_o = -\frac{Q}{8\pi^2} \iint_{[-\infty, \infty; -\infty, \infty]} \frac{d \ln h(\eta)}{d\eta} \frac{\eta}{\xi^2 + \eta^2} \ln[(\xi - x)^2 + (\eta - y)^2] d\xi d\eta. \quad (6)$$

The region of integration for the double integral in this and subsequent formulas is the (x, y) plane.

It is easy to show that

$$\frac{\partial\phi'_o}{\partial y} = -\frac{Q}{8\pi} \frac{d \ln h}{dy} \ln \left[\frac{x^2 + y^2}{x^2 + (y+a)^2} \right] + f(x, y), \quad (7)$$

where $f(x, y)$ is bounded everywhere and $f(x, y) \rightarrow 0$ as $r \rightarrow \infty$.

Thus we can write the solution of equation (5) in the form of an integral equation

$$\begin{aligned} \phi''_o(x, y) &= -\frac{1}{4\pi} \iint_{[-\infty, \infty; -\infty, \infty]} \frac{d \ln h(\eta)}{d\eta} \frac{\partial\phi_o(\xi, \eta)}{\partial\eta} \times \\ &\quad \times \ln[(\xi - x)^2 + (\eta - y)^2] d\xi d\eta. \end{aligned} \quad (8)$$

With the method of iterations let us show that for laminae with $h_{-\infty}/h_{\infty} < e^2$ there exists a solution of equation (8) bounded in the entire plane and tending to zero as $r \rightarrow \infty$.

Successive approximations are determined in the following way:

$$\begin{aligned} \frac{\partial \phi''_{o_i}(x, y)}{\partial y} &= \frac{1}{2\pi} \iint_{[-\infty, \infty; -\infty, \infty]} \frac{d \ln h(\eta)}{d\eta} \times \\ &\times \left[\frac{\partial \phi'_o(\xi, \eta)}{\partial \eta} + \frac{\partial \phi''_{o_{(i-1)}}(\xi, \eta)}{\partial \eta} \right] \frac{(\eta - y)}{(\xi - x)^2 + (\eta - y)^2} d\xi d\eta, \end{aligned} \quad (9)$$

where $\partial \phi''_{o_0}/\partial y = 0$. From the equations (7) and (9) it follows that $\partial \phi''_{o_i}/\partial y$ are everywhere bounded and $\lim_{r \rightarrow \infty} \partial \phi''_{o_i}/\partial y = 0$, since $\lim_{r \rightarrow \infty} \partial \phi'_o/\partial y = 0$,

Let us prove the convergence of the iterative process. Writing the equation (8) in operator form we get $t = A(t)$, where $t = \partial \phi'_o/\partial y$. Consider the metric space of all approximations t_i ($i = 1, 2, \dots, \infty$) with the norm $\rho(t_j, t_k) = \max |t_j - t_k|$. Then

$$\begin{aligned} \rho(A(t_j), A(t_k)) &= \max \left| \frac{1}{2\pi} \iint_{[-\infty, \infty; -\infty, \infty]} (t_j - t_k) \frac{d \ln h(\eta)}{d\eta} \times \right. \\ &\quad \left. \times \frac{(\eta - y)}{(\xi - x)^2 + (\eta - y)^2} d\xi d\eta \right|, \end{aligned}$$

i. e.

$$\rho(A(t_j), A(t_k)) \leq \frac{1}{2} \rho(t_j, t_k) \left| \int_{-\infty}^{\infty} \frac{d \ln h(\eta)}{d\eta} y d\eta \right|,$$

since

$$\int_{-\infty}^{\infty} \frac{(\eta - y)}{(\xi - x)^2 + (\eta - y)^2} d\xi \leq \pi,$$

or, finally,

$$\rho(A(t_j), A(t_k)) \leq \frac{1}{2} \ln \frac{h_{(-\infty)}}{h_{(\infty)}} \rho(t_j, t_k).$$

By the principle of contraction mappings [2] the iterative process converges, if $h_{(-\infty)}/h_{(\infty)} < e^2$. That means that the solution (8) is a solution to (5) with the boundary conditions $\nabla \phi''_o|_{r \rightarrow \infty} = 0$, since $\lim_{r \rightarrow \infty} \partial \phi''_o/\partial x = 0$.

III. Now we obtain the boundary conditions for the flow due to an infinite sequence of sources satisfying (1), by summing the velocities due to the concentrated sources of this sequence and taking the limit of the sum as $y \rightarrow \pm\infty$. Here is a partial sum

$$\begin{aligned} s_m &= \sum_{n=-m}^m \frac{\partial \phi_{o_n}}{\partial y} = \frac{1}{2\pi} \sum_{n=-m}^m \iint_{[-\infty, \infty; -\infty, \infty]} \frac{d \ln h(\eta)}{d\eta} \frac{\partial \Phi_o(\xi - nt, \eta)}{\partial \eta} \times \\ &\quad \times \frac{(\eta - y)}{(\xi - x)^2 + (\eta - y)^2} d\xi d\eta \end{aligned}$$

or, if we change variables $\xi_n = \xi - nt$ and denote ξ_n by ζ ,

$$s_m = \frac{1}{2\pi} \left[\sum_{n=-m}^m \iint_{[-M, M; -N, N]} \frac{d \ln h(\eta)}{d\eta} \frac{\partial \Phi_o(\zeta, \eta)}{\partial \eta} \times \right.$$

$$\times \frac{(\eta - y)}{(\zeta + nt - x)^2 + (\eta - y)^2} d\zeta d\eta + R \Big].$$

Since $\lim_{r \rightarrow \infty} \partial\Phi_o/\partial\eta = 0$, it can be shown that $R = R(x, y, m, M, N) < \epsilon(M, N)$, where $\epsilon(M, N) \rightarrow 0$ as $M \rightarrow \infty$ and $N \rightarrow \infty$ uniformly with respect to x, y and m for $|x| \leq t/2, N < |y| < \infty$ and $0 < m < \infty$. If we let m tend to infinity and apply the theorem on integration of a uniformly convergent series, we get

$$\begin{aligned} \frac{\partial\phi_s}{\partial y} &= \frac{1}{2\pi} \left[\int_{-N}^N d\eta \int_{-M}^M \frac{d \ln h(\eta)}{d\eta} \frac{\partial\Phi_o(\zeta, \eta)}{\partial\eta} \times \right. \\ &\quad \left. \times \frac{\sinh \frac{2\pi}{t}(\eta - y)}{\cosh \frac{2\pi}{t}(\eta - y) - \cos \frac{2\pi}{t}(\zeta - x)} d\zeta + R \right], \end{aligned}$$

or, finally,

$$\lim_{y \rightarrow \infty} \frac{\partial\phi_s}{\partial y} = - \lim_{y \rightarrow -\infty} \frac{\partial\phi_s}{\partial y} = -\frac{1}{2t} \iint_{[-\infty, \infty; -\infty, \infty]} \frac{d \ln h(\eta)}{d\eta} \frac{\partial\Phi_o(\zeta, \eta)}{\partial\eta} d\zeta d\eta.$$

Now taking into account that

$$\left[\frac{\partial\Phi_{s_0}}{\partial y} \right]_{y=\infty} = - \left[\frac{\partial\Phi_{s_0}}{\partial y} \right]_{y=-\infty},$$

we obtain the conditions (3).

IV. In order to obtain a solution of the equation(1) in closed form, let us apply the Fourier transform with infinite limits to the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\frac{d \ln h}{dy} \left[\frac{\partial\Phi_0}{\partial y} + \frac{\partial\phi}{\partial y} \right], \quad (10)$$

where $\Phi = \Phi_0 + \phi$ is the potential of a source and a sink located at the points $(0, 0)$ and $(x_0, 0)$ respectively. The introduction of a sink is necessary, because the Fourier transform with infinite limits cannot be applied to a source alone, since its potential $|\Phi_o| \rightarrow \infty$ as $r \rightarrow \infty$. Having solved equation (10) we let x_0 tend to infinity and obtain an expression for $\partial\Phi_o/\partial x$ and $\partial\Phi_o/\partial y$. Then we obtain $\partial\Phi_s/\partial x$ and $\partial\Phi_s/\partial y$ by summing.

Transformed equation (10)

$$\frac{d^2\bar{\phi}}{dy^2} + \frac{d \ln h}{dy} \frac{d\bar{\phi}}{dy} - \zeta^2\bar{\phi} = \frac{Q}{2\sqrt{2\pi}} [e^{i\zeta x_0} - 1] \frac{d \ln h}{dy} y \frac{e^{-|\zeta y|}}{|y|}$$

with boundary conditions $\bar{\phi}|_{y=\pm\infty} = 0$ cannot be solved by quadratures for laminae occurring in practice. Therefore, we will solve the equation (10) with the method of successive approximations (6) and (9) applying each time the Fourier transform. As shown above, the iterative process converges for laminae

with $h_{(-\infty)}/h_{(\infty)} < e^2$ (this condition is satisfied by laminae of a flow in a radial-axial hydroturbine). We can find ϕ' from the equation

$$\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} = -\frac{d \ln h}{dy} \frac{\partial \Phi_0}{\partial y},$$

by taking the Fourier transform. Letting x_0 go to infinity we obtain $\frac{\partial \phi'_o}{\partial x}$ and $\frac{\partial \phi'_o}{\partial y}$, and then $\frac{\partial \phi'_s}{\partial x}$ and $\frac{\partial \phi'_s}{\partial y}$.

To solve equation (10) we assumed the following approximation for the thickness of the lamina respectively for $y \geq 0$ and $y \leq 0$:

$$\begin{aligned} \ln h^{(+)} &= \ln h_{(\infty)} + \sum_{k=1}^{l_1} A_k e^{-B_k y}, \\ \ln h^{(-)} &= \ln h_{(-\infty)} + \sum_{k=1}^{l_2} C_k e^{-D_k y}, \end{aligned}$$

where $B_k > 0, D_k > 0$. Note that $\ln h_0^{(+)} = \ln h_0^{(-)}$ and $d \ln h^{(+)}(0)/dy = d \ln h^{(-)}(0)/dy$.

We show the formula for $\partial \phi'_s / \partial y$ for $y \geq 0$:

$$\begin{aligned} \frac{\partial \phi'_s}{\partial y} &= \frac{Q}{4t} \left[\ln \left(\frac{h_{(0)}}{h} \right) \frac{\sinh \frac{2\pi}{t} y}{\cosh \frac{2\pi}{t} y - \cos \frac{2\pi}{t} x} + \right. \\ &+ \sum_{k=1}^{l_1} \sum_{n=1}^m \alpha_n A_k e^{-B_k y} \frac{\sinh \frac{2\pi}{t} (y + \frac{2\beta_n}{B_k})}{\cosh \frac{2\pi}{t} (y + \frac{2\beta_n}{B_k}) - \cos \frac{2\pi}{t} x} + \\ &+ \sum_{k=1}^{l_2} \sum_{n=1}^m \alpha_n C_k \frac{\sinh \frac{2\pi}{t} (y + \frac{2\beta_n}{D_k})}{\cosh \frac{2\pi}{t} (y + \frac{2\beta_n}{D_k}) - \cos \frac{2\pi}{t} x} + \\ &\left. + \frac{t}{\pi} \sum_{k=1}^{l_1} \sum_{n=1}^m \alpha_n A_k B_k e^{-B_k y} \ln \left(\frac{\cosh \frac{2\pi}{t} y - \cos \frac{2\pi}{t} x}{\cosh \frac{2\pi}{t} (y + \frac{2\beta_n}{B_k}) - \cos \frac{2\pi}{t} x} \right) \right], \quad (11) \end{aligned}$$

where $\alpha_1, \alpha_2, \dots, \alpha_m$ and $\beta_1, \beta_2, \dots, \beta_m$ are determined from the expansion

$$\frac{1}{1+x} = \frac{1}{x} - \sum_{n=1}^m \frac{1}{x} \alpha_n e^{-\beta_n x} \quad \text{for } x \geq 0.$$

From equation (11) it can be seen that

$$\left[\frac{\partial \phi'_s}{\partial y} \right]_{y=\infty} = \left[\frac{\partial \phi'_s}{\partial y} \right]_{y=-\infty} = \frac{Q}{2t} \ln \left(\frac{h_{(0)}}{\sqrt{h_{(\infty)} h_{(-\infty)}}} \right).$$

The final solution for the potential due to a concentrated source can be written in the form

$$\Phi_o = \Phi_{o0} + \phi'_o + \phi''_{o1} + \sum_{j=2}^{\infty} \Delta \phi''_{oj},$$

where $\Delta\phi''_{o_j}$ for $j > 2$ are determined from the equations

$$\frac{\partial^2 \Delta\phi''_{o_j}}{\partial x^2} + \frac{\partial^2 \Delta\phi''_{o_j}}{\partial y^2} = -\frac{d \ln h}{dy} \frac{\partial \phi''_{o_{(j-1)}}}{\partial y}.$$

V. Analogous formulas for vortices can be obtained from the equation

$$\frac{\partial^2 \psi_v}{\partial x^2} - \frac{d \ln h}{dy} \frac{\partial \psi_v}{\partial y} + \frac{\partial^2 \psi_v}{\partial y^2} = \Gamma \delta(x) \delta(y). \quad (12)$$

Since $V_{x\Gamma} = \frac{1}{h} \frac{\partial \psi_v}{\partial y}$, the conditions at infinity will be $[hV_{x\Gamma}]_{y=\infty} = -[hV_{x\Gamma}]_{y=-\infty}$.

VI. For laminae with $h = h_1(x)h_2(y)$ the solution can be obtained as a sum of three solutions $\Phi_o = \Phi_{o_0} + \phi_{o_{x_1}} + \phi_{o_{y_2}}$, where Φ_{o_0} is the potential of the problem of a lamina of constant thickness and $\phi_{o_{x_1}}$ and $\phi_{o_{y_2}}$ are additional potentials caused by the variability of thickness along the x and y axes. Clearly, according to a theorem of Farr and Keen [**3**], the above calculations can be applied to all laminae that can be conformally mapped to the considered cases.

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References

- [**1**] S. V. Vallander, Doklady Akademii Nauk SSSR, **123**, N^o3 (1958).
- [**2**] L. A. Lusternik, S. L. Sobolev, *Elements of functional analysis*, 1951.
- [**3**] H. K. Farr, W. A. Keen, Comm. Electronics, **7**, N^o19 (1955).