On the quadrature of a circle and circulature of a square in Lobachevskij space

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§ **1.** The following remarks are directed towards the solution of the posed problem.

1° In Lobachevskij space a side a_n of a regular n-gon inscribed in a circle of radius R is given by the equation:

$$\sinh \frac{a_n}{2k} = \sinh \frac{R}{k} \sin \pi n.$$

Since in the space L constructions by means of the complex E (ruler and compass) of 2nd order and any class are possible according to a theorem of D.D. Morduchai-Boltovskoj [1], the problem of constructing a regular n-gon is solvable by E in all cases when $\sinh \pi/n$ is an expression constructed from 1 by means of a finite number of rational operations and extraction of square roots. Therefore, the problem of dividing a circle into n equal parts in the space L is solvable with the complex E for the same cases as in the space E, namely exactly when

$$n = 2^r p_1 p_2 \ldots p_m, \qquad (1)$$

where $r = 0, 1, 2, \ldots$ and p_i are prime numbers of the form $2^{2^s} + 1$.

2° In the space L it is possible to construct with the complex E a right triangle from its two acute angles [2].

3° In the space L it is possible to construct with the complex E a square from its acute angle α, since $1/8$ of the square is a right triangle with acute angles $\alpha/2$ and $\pi/4$.

§ **2.** A special case of the theory of areas in the space L leads to the fact that the problems of quadrature of a circle and circulature of a square are algebraic rather than transcendental in infinitely many cases.

§ 3. **Circulature of a square.** From equation $S_\circ = S_\Box$ or from

$$4\pi k^2 \sinh^2 \frac{R}{2k} = k^2 \left(2\pi - \Sigma_\Box\right) \tag{2}$$

it follows that the considered problem, i.e. the construction of R is solvable by means of the complex E when the sum of the internal angles of the square $\Sigma_\Box$ is equal to $\pi\omega$, where ω is a rational number and is constructed from 1 by means of a finite number of rational operations and extraction of square roots.

In this case equation (2) takes the following form:

$$\sinh \frac{R}{2k} = \sqrt{\frac{1}{2} - \frac{\omega}{4}}, \qquad 0 \leq \omega < 2.$$

The case of Bolyai corresponds to the value $\omega = 1$.

§ 4. **Quadrature of a circle.** Since a square is determined constructively by its acute angle α (§ 1, 3°) and from equation (2) we find that

$$\alpha = \pi - \frac{\pi}{2} \cosh \frac{R}{k},$$

where

$$1 < \cosh R/k \leq 2, \tag{3}$$

the solvability of the problem of quadrature of a circle by means of complex E depends on the possibility of constructing by these means the angle

$$\beta = \frac{\pi}{2} \cosh \frac{R}{k}.$$

For this to happen $\cosh R/k$ must be a not reduced rational fraction $\frac{m+n}{n}$ within the limits (3) and the denominator n must be a number of the form (1).

Thus, the problem of quadrature of a square in the space L is equivalent to the algebraic problem of dividing a circle, and is therefore solvable by means of the complex E is an infinite number of cases, when the angle β is constructible. The given values of β exhaust all possible cases of solvability of the problem of quadrature of a circle by means of the complex E.

§ 5. The class of circulable squares is wider than the class of quadrable circles. For example, the square with angle

$$\alpha = \frac{\pi\sqrt{2}}{4}$$

is circulable and gives a circle, whose radius R, is given by the formula

$$\sinh \frac{R}{2k} = \frac{\sqrt{2 - \sqrt{2}}}{2}.$$

However quadrature of the circle of radius R, given by the formula, is impossible by means of E (ruler and compass), since the angle (4) does not belong the class of angles, defined by means of the numbers (1).

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