

COMPARABILITY CLASSES AND
THE SIGN OF A DIFFERENTIAL POLYNOMIAL
OF A LARGE FUNCTION IN A HARDY FIELD

Definition 0.1 Let K be a Hardy field with the canonical valuation $\nu: K^* \rightarrow \Gamma$. (see [R1]). For $f \in K$ such that $f \neq 0$ and $\nu(f) \neq 0$ define the *comparability class* of f to be

$$Cl(f) = \{g \in K : m|\nu(f)| > |\nu(g)| \text{ and } |\nu(f)| < n|\nu(g)| \text{ for some } m, n \in \mathbf{Z}^+\}.$$

Remark 0.1 The set of comparability classes of K is linearly ordered in an obvious way.

Remark 0.2 Suppose $f, g \in K; f, g \neq 0; \nu(f), \nu(g) \neq 0$ and $Cl(f) > Cl(g)$. Then $Cl(fg) = Cl(f)$ [R2].

Definition 0.2 For $f \in K$ set $\lambda_0(f) = f$ and if $f \neq 0$, define $\lambda_1(f) = \lambda(f) = (\ln f)' = f'/f$. If $f = 0$, define $\lambda(f) = 0$. For $i > 1$ inductively define $\lambda_i(f) = \lambda(\lambda_{i-1}(f))$ [B].

Remark 0.3 Clearly for all $f \in K$ and all $i \in \mathbf{Z}^+$, $\lambda_i(f) \in K$.

Proposition 0.1 Let K be a Hardy field containing $f, g \neq 0$ such that $\nu(f), \nu(g) < 0$. Then $Cl(f) \leq Cl(g)$ if and only if $\nu(\lambda(f)) \geq \nu(\lambda(g))$.

Proof: Without loss of generality assume that $f, g > 0$ and for all non-zero $h \in K, \ln |h| \in K$. Suppose $\nu(\lambda(f)) \geq \nu(\lambda(g))$. By l'Hôpital's rule this is equivalent to $\nu(\ln(f)) \geq \nu(\ln(g))$ (see theorem 4 [R1]), i.e.

$$\lim_{x \rightarrow +\infty} \frac{\ln f}{\ln g} = c \in \mathbf{R}.$$

Thus, for all $\epsilon > 0, c - \epsilon < \ln f / \ln g < c + \epsilon$. Since $\nu(f), \nu(g) < 0$, we ultimately have $\ln f, \ln g > 0$, so $c \geq 0$ and $\ln(g^{c-\epsilon}) < \ln f < \ln(g^{c+\epsilon})$. Therefore, $g^{c-\epsilon} < f < g^{c+\epsilon}$. If $c > 0$, then $Cl(f) = Cl(g)$ and if $c = 0$, then $Cl(f) < Cl(g)$.

Conversely, assume that $Cl(f) \leq Cl(g)$. Then there exists an $n \in \mathbf{Z}^+$ such that $f < g^n$, i.e. $\ln f < n \ln g$. Then

$$0 < \frac{\ln f}{\ln g} < n$$

and we can reverse the previous argument. ■

Proposition 0.2 Let K be a Hardy field containing $f \neq 0$ such that $\nu(f) < 0$ and $Cl(f) > Cl(x)$. Then $Cl(\lambda(f)) < Cl(f)$.

Proof: Without loss of generality assume that $f > 0$. Let $\epsilon > 0$ and $a \in \mathbf{R}$ sufficiently large. Then

$$\int_a^{+\infty} \frac{f'(t)}{f^{1+\epsilon}(t)} dt = \lim_{x \rightarrow +\infty} \int_a^{+\infty} \frac{f'(t)}{f^{1+\epsilon}(t)} dt = -\frac{1}{\epsilon} \left(\lim_{x \rightarrow +\infty} f^{-\epsilon}(x) - f^{-\epsilon}(a) \right) = \frac{1}{\epsilon} f^{-\epsilon}(a),$$

because $\nu(f) < 0$. Since the integral converges, ultimately $|f'/f^{1+\epsilon}| < 1$, so $|\lambda(f)| < f^\epsilon$ for all $\epsilon > 0$. If $\nu(\lambda(f)) \leq 0$, then we are done, so assume that $\nu(\lambda(f)) > 0$. By proposition 0.1, since $Cl(f) > Cl(x)$, $\nu(\lambda(f)) < \nu(\lambda(x)) = \nu(1/x)$, i.e. $\nu(f/f') > \nu(x)$. Therefore, for sufficiently large x , $|f/f'| > x$, so $Cl(\lambda(f)) = Cl(f/f') \leq Cl(x) < Cl(f)$. ■

Proposition 0.3 Let K be a Hardy field containing $f \neq 0$ such that $\nu(f) < 0$ and $Cl(f) > Cl(x)$. Then $\nu f' < 0$ and $Cl(f') = Cl(f)$.

Proof: Observe that $\nu(f/x) < 0$, so by l'Hôpital's rule, $\nu(f') < 0$. By proposition 0.2, $Cl(f) > Cl(\lambda(f))$, so by remark 0.2, $Cl(f) = Cl(f\lambda(f)) = Cl(f')$. ■

Corollary Let K be a Hardy field containing $f \neq 0$ such that $\nu(f) < 0$ and $Cl(\ln|f|) > Cl(x)$. Then $\nu(\lambda(f)) < 0$ and $Cl(\lambda(f)) = Cl(\ln|f|)$.

Remark 0.4 If $K \subseteq H$ are Hardy fields and $f \in H$, then the field extensions of K in H generated by $\{f, f', \dots, f^{(k)}\}$ and $\{\lambda_0(f), \dots, \lambda_k(f)\}$ are identical. In fact,

$$f = \lambda_0(f), f' = \lambda_1(f)\lambda_0(f), f'' = \lambda_2(f)\lambda_1(f)\lambda_0(f) + (\lambda_1(f))^2 \lambda_0(f), \dots$$

and an easy induction argument shows that for all $i \in \mathbf{Z}^+$, $f^{(i)}$ is a polynomial in $\lambda_j(f)$, $j \in \{0, 1, \dots, i\}$.

Furthermore, if $p \in K[X_0, X_1, \dots, X_k]$, then $p(f, f', \dots, f^{(k)})$ can be expressed as a polynomial in $\lambda_i(f)$, $i \in \{0, 1, \dots, k\}$, which will be denoted by q_p . Note that if p is non-trivial, then so is q_p [B].

Lemma Let K be a Hardy field containing $f, g \neq 0$ such that $\nu(f), \nu(g) < 0$ and $Cl(f) > Cl(g)$. Then $Cl(e^f) > Cl(e^g)$.

Proof: Let $0 < \epsilon < 1$. Then $g < f^\epsilon$ and $e^g < e^{f^\epsilon}$. Since $\nu(f) < 0$, $\nu(f^\epsilon/f^\epsilon) = \nu(f)(1 - \epsilon) < 0$, so eventually $f^\epsilon < f\epsilon$ and $e^g < (e^f)^\epsilon$. ■

Theorem (see theorem 12.15 [B]) Let $K \subseteq H$ be Hardy fields such that for all $g \in K$, $Cl(g) \leq Cl(x)$. Let $f \in H$ such that $f > 0$, $\nu(f) < 0$ and $Cl(\ln_k f) > Cl(x)$ for some $k \in \mathbf{Z}^+$. Then for any $p \in K[X_0, X_1, \dots, X_k]$, the sign of $p(f, f', \dots, f^{(k)})$ is independent of the choice of f and can be explicitly determined.

Proof: The lemma and $Cl(\ln_k f) > Cl(x)$ imply that for all $i \in \{0, 1, \dots, k\}$, $Cl(\ln_i f) > Cl(e_{k-i}(x))$. Applying the corollary and proposition 0.2 successively to $\lambda_i(f)$, $i \in \{0, 1, \dots, k-1\}$ we see that

$$Cl(f) > Cl(\lambda(f)) = Cl(\ln f) > \dots > Cl(\lambda_k(f)) = Cl(\ln_k f) > Cl(x).$$

By remark 0.4, we can replace $p(f, f', \dots, f^{(k)})$ with $q_p(\lambda_0(f), \lambda_1(f), \dots, \lambda_k(f))$ and by the above formula and remark 0.2, q_p has a dominant term, which can be determined alphanumerically. ■

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