Vector interpretations of exterior calculus in $\mathbf{R}^3$

Vector interpretation of $\wedge$ and d:

[0]: Let f and g be 0-forms (f and g are scalar functions)

[1]: Let $a \cdot dr$ and $b \cdot dr$ be 1-forms (a and b are vector fields and $dr = (dx, dy, dz)$)

[2]: Let $p \cdot ds$ and $q \cdot ds$ be 2-forms (p and q are vector fields and $ds = (dy\,dz, dz\,dx, dx\,dy)$)

[1,1]: $(a \cdot dr) \wedge (b \cdot dr) = (a \times b) \cdot ds$

[1,2]: $(a \cdot dr) \wedge (p \cdot ds) = (a \cdot p)\,dx\,dy\,dz$

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[0]: $df = (\nabla f) \cdot dr$

[1]: $d(a \cdot dr) = (\nabla \times a) \cdot ds$

[2]: $d(p \cdot ds) = (\nabla \cdot p)\,dx\,dy\,dz$

Product rule $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^{\deg \omega} \omega \wedge d\theta$:

[0,0]: $\nabla(fg) = (\nabla f)g + f(\nabla g)$

[0,1]: $\nabla \times (fa) = \nabla f \times a + f(\nabla \times a)$

[0,2]: $\nabla \cdot (fp) = \nabla f \cdot p + f(\nabla \cdot p)$

[1,1]: $\nabla \cdot (a \times b) = (\nabla \times a) \cdot b - a \cdot (\nabla \times b)$

Vector interpretation of $d^2 = 0$:

[0]: $\nabla \times (\nabla f) = 0$

[1]: $\nabla \cdot (\nabla \times a) = 0$

Fundamental theorem of calculus $\int_\Omega d\omega = \int_{\partial\Omega} \omega$:

[0]: $\int_{\mathscr{L}} \nabla f \cdot d\ell = f$ evaluated at the endpoints of $\mathscr{L}$ (Barrow's rule)

[1]: $\int_{\mathscr{D}} \nabla \times a\,ds = \oint_{\partial\mathscr{D}} a\,d\ell$ (Stokes-Kelvin)

[2]: $\int_{\Omega} \nabla \cdot p\,ds = \int_{\partial\Omega} p\,dV$ (Gauss-Ostrogradski)