**Remainder of the geometric series:**

Let $S_n(z) = \sum_{k=0}^{n} z^k$ be the $n$-th partial sum of the geometric series.

Multiplying and cancelling we obtain $(1-z)S_n(z) = 1 - z^{n+1}$, so $S_n(z) = \dfrac{1-z^{n+1}}{1-z}$

If $|z| < 1$, then $z^{n+1} \to 0$ as $n \to \infty$, so $S_n(z) \to \dfrac{1}{1-z}$

Writing $\dfrac{1}{1-z} = S_n(z) + R_n(z)$ we see that the remainder is $R_n(z) = \dfrac{z^{n+1}}{1-z}$

**Taylor remainder:**

Supoose $f \in \mathcal{H}(\Omega)$, where $\Omega$ is a domain in $\mathbf{C}$. If $a \in \Omega$, we can find an open disc $D$ of radius $r$ centered at $a$ with $\overline{D} \subseteq \Omega$. From the values of $f$ on $\partial D$ Cauchy's integral formula gives the values of $f$ for all $z \in D$

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)\, d\zeta}{\zeta - z} \tag{1}$$

Now use the partial geometric sum to obtain

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a)\left(1 - \dfrac{z-a}{\zeta - a}\right)} = \frac{1}{\zeta - a}\left(\sum_{k=0}^{n}\left[\frac{z-a}{\zeta-a}\right]^k + \frac{\left[\dfrac{z-a}{\zeta-a}\right]^{n+1}}{1 - \dfrac{z-a}{\zeta-a}}\right) = \sum_{k=0}^{n}\frac{(z-a)^k}{(\zeta-a)^{k+1}} + \frac{\left[\dfrac{z-a}{\zeta-a}\right]^{n+1}}{\zeta - z}$$

Substitute this finite sum into Cauchy's integral formula to obtain $f(z) = \sum_{k=0}^{n} c_k (z-a)^k + R_n(z)$ where

$$c_k = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\, d\zeta}{(\zeta - a)^{k+1}}, \qquad R_n(z) = \left[\frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\, d\zeta}{(\zeta-a)^{n+1}(\zeta - z)}\right](z-a)^{n+1}$$

Successive differentiation of (1) under the integral sign with respect to $z$ gives $c_n = \dfrac{f^{(n)}(a)}{n!}$

**Cauchy's intequalities and convergence:**

Since $\overline{D}$ is compact, $|f|$ is bounded on $\overline{D}$. Suppose $|f| \leq M$ on $\partial D$. Then

$$|c_k| \leq \frac{1}{2\pi}\int_{\partial D} \frac{|f(\zeta)|}{r^{k+1}}\, |d\zeta| \leq \frac{M}{2\pi r^{k+1}} \int_{\partial D} |d\zeta| = \frac{M}{2\pi r^{k+1}} \cdot 2\pi r = \frac{M}{r^k}$$

Similarly, since $|\zeta - z| = |(\zeta - a) - (z - a)| \geq |\zeta - a| - |z - a|$

$$|R_n(z)| \leq \frac{M}{r^n} \cdot \frac{(z-a)^{n+1}}{r - |z-a|} = \frac{M\,|z-a|}{r - |z-a|}\left|\frac{z-a}{r}\right|^n \to 0 \text{ as } n \to \infty$$

This estimate for the remainder shows that the Taylor series converges on $D$. Note that the only restriction on $D$ is that $\overline{D} \subset \Omega$ so the radius of convergence is the distance from $a$ to the boundary of $\Omega$.