Normality of metric spaces and the shrinking lemma

Definition: A topological space is normal (a.k.a. T_4) whenever given two disjoint closed subsets A, B, there exist disjoint open subsets U, V such that $A \subseteq U, B \subseteq V$.

Theorem: If (X, d) is a metric space, then it is normal.

Proof: Given $S \subseteq X$, define $f_S\colon X \to \mathbf{R}$ by $f_S(x) = d(x, S) \stackrel{\text{def}}{=} \inf\{d(x, a)\colon a \in S\}$. Then $f_S \geq 0$, f_S is continuous and $f_S(x) = 0 \Leftrightarrow x \in \overline{S}$. In particular, if S is closed, then $S = f_S^{-1}(\{0\})$.
Let $f = f_A/(f_A + f_B)$. Then $0 \leq f \leq 1$. Since A and B are disjoint the denominator is never zero, so f is continuous. Furthermore $f \equiv 0$ on A and $f \equiv 1$ on B. Let $U = f^{-1}((-\infty, 1/2))$ and $V = f^{-1}((1/2, \infty))$. ∎

Note: A function with the properties of f is called a Urysohn function after Pavel Urysohn (b. Odessa 1898, drowned off the coast of Bretagne 1924). The existence of a Urysohn function clearly implies normality. Urysohn's lemma (4.3.1 [2], VII.4.1 [1]) proves the existence of such a function for any two disjoint closed subsets of an arbitrary normal space. Munkres calls it the first deep theorem of his book.

Lemma: Suppose X is normal, and $\{U, V\}$ is an open cover of X. Then there exists an open set W such that $\overline{W} \subseteq U$ and $\{W, V\}$ is still an open cover of X.

Proof: Since $X = U \cup V$, $U^c \cap V^c = \emptyset$. By normality there exist disjoint open sets S, W such that $U^c \subseteq S, V^c \subseteq W$. Then $S^c \subseteq U$ and $W^c \subseteq V$, so in particular $X = W \cup V$. Since $W \cap S = \emptyset$, $W \subseteq S^c$. Thus, $\overline{W} \subseteq S^c \subseteq U$. ∎

Shrinking Lemma: Suppose X is normal, and $\{U_i\colon i = 1, ...n\}$ is a finite open cover of X. Then there exist open sets W_i such that $\overline{W_i} \subseteq U_i$ and $\{W_i\colon i = 1, ...n\}$ is still an open cover of X.

Proof: Induction on the preceding lemma. ∎

Note: The shrinking lemma can be generalized to an infinite cover as long as it is *point-finite*, i.e. each point of X is in at most finitely many covering sets. The proof is by transfinite induction. This shrinking property is equivalent to normality (VII.6.1 [1]).

References:

[1] J. Dugundji, *Topology*, Allyn and Bacon, 1966

[2] J. Munkres, *Topology: a first course*, Prentice-Hall, 1975