Projection and dot product

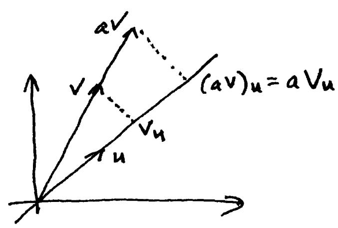
Fix a vector $u = u_x\widehat{\imath} + u_y\widehat{\jmath} \neq 0$. Given a vector $v = v_x\widehat{\imath} + v_y\widehat{\jmath}$ we consider the orthogonal projection of v to u (component of v along u). If θ is the angle between v and u, then from geometrical considerations this projection is

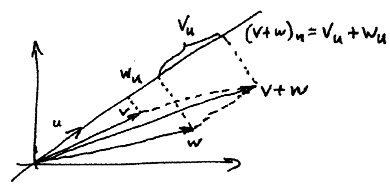
$$v_u = |v| \cos\theta$$

Theorem.

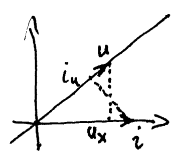
$$v_u = \frac{u \cdot v}{|u|} = \frac{u_x v_x + u_y v_y}{\sqrt{u_x^2 + u_y^2}}$$

Proof. From geometrical considerations it follows that the map $v \mapsto v_u$ is linear:





so if $v = v_x\widehat{\imath} + v_y\widehat{\jmath}$, then $v_u = \left(v_x\widehat{\imath} + v_y\widehat{\jmath}\right)_u = v_x\widehat{\imath}_u + v_y\widehat{\jmath}_u$.
Assuming $|u| = 1$, from geometrical considerations:



we see that $\widehat{\imath}_u = u_x$ and $\widehat{\jmath}_u = u_y$, so $v_u = v_x u_x + v_y u_y$. If $|u| \neq 1$, replace u by $u/\,|u|$. QED

Corollary.

$$u \cdot v = u_x v_x + u_y v_y = |u|\,|v| \cos\theta$$

Properties of dot product

1. bi-linear (linear in each variable): $(au+bv)\cdot w = a(u\cdot w)+b(v\cdot w)$, $u\cdot(av+bw) = a(u\cdot v)+b(u\cdot w)$,
2. symmetric: $u \cdot v{=}v \cdot u$,
3. positive definite: $u \cdot u \geq 0$ and $u \cdot u = 0 \Rightarrow u = 0$.