Multinomial Formula by Dr. Dmitry Gokhman 1995

$$(a_1 + ... + a_m)^n = \sum_{i_1+...+i_m=n} C(n, i_1, ...i_m)\, a_1^{i_1} ... a_m^{i_m},$$

where we assume $i_k \geq 0$ and the multinomial coefficients are

$$C(n, i_1, ...i_m) = \frac{n!}{i_1!...i_m!}.$$

Using multi-index notation (e.g. $I = (i_1, ...i_m)$, $|I| = \sum_{k=1}^{m} i_k$) the multinomial formula can written more compactly

$$\left(\sum_{k=1}^{m} a_k\right)^n = \sum_{|I|=n} C(n, I) \prod_{k=1}^{m} a_k^{i_k}, \qquad C(n, I) = \frac{n!}{\prod_{k=1}^{m} i_k!}.$$

Example

The binomial formula is a special case

$$(a + b)^n = \sum_{i+j=n} C(n, i, j)\, a^i b^j, \quad C(n, i, j) = \frac{n!}{i!\, j!}$$

and one can obtain a more familiar form of this by substituting $j = n - i$ and omitting j from the coefficient

$$(a + b)^n = \sum_{i=0}^{n} C(n, i)\, a^i b^{n-i}, \quad C(n, i) = \frac{n!}{i!\, (n - i)!}.$$

Recursion for the coefficients

Binomial coefficients are often calculated by a "Pascal triangle" recursion:

$$C(n, i, j) = C(n - 1, i - 1, j) + C(n - 1, i, j - 1)$$

or in more conventional notation with j dropped

$$C(n, i) = C(n - 1, i - 1) + C(n - 1, i).$$

For multinomial coefficients there is a recursion corresponding to a higher dimensional object (triangle $\rightarrow$ tetrahedron $\rightarrow$ etc.):

$$C(n, i_1, ...i_m) = C(n-1, i_1-1, i_2, ...i_m) + C(n-1, i_1, i_2-1, ...i_m) + ... + C(n-1, i_1, i_2, ...i_m-1)$$