Intermediate Value Theorem

Holy Intermediate Value Theorem, Batman! They must have crossed the road somewhere.

This is an important topological result often used in establishing existence of solutions to equations. It says that a continuous function attains all values between any two values. A key ingredient is completeness of the real line.

Theorem (IVT): Suppose $f\colon [a,b] \to \mathbf{R}$ is continuous and c is between $f(a)$ and $f(b)$. Then there exists s between a and b such that $f(s) = c$.

Proof: Without loss of generality we may assume $f(a) < c < f(b)$. Let $S = \{x \in [a,b]\colon f(x) < c\}$. Since $a \in S$, S is nonempty, so since S is bounded above, by completeness of $\mathbf{R}$, S has a supremum s. Since any neighborhood of s contains points of both S and its complement (i.e. points where f is greater and smaller than c) and f is continuous at s, $f(s) = c$.

Babylonian bisection: Another proof can be obtained constructively as follows. Again assume $f(a) < c < f(b)$. Let $I_1 = [a,b]$ and let x_1 be the midpoint of I_1. If $f(x_1) = c$ we are done. If $f(x_1) < c$ let $I_2 = [x_1, b]$. Otherwise let $I_2 = [a, x_1]$ and proceed by induction. If we never stop, let (a_i) and (b_i) be the sequences of left and right endpoints of I_i. Then

(a) (a_i) is increasing and (b_i) is decreasing
(b) $f(a_i) < c < f(b_j)$
(c) $I_0 \supset I_1 \supset \ldots,$
(d) $b_i - a_i = 2^{-i}(b-a)$

By (a) and (c), (a_i) is monotone and bounded, so has a limit s. Since f is continuous at s, we have $f(a_i) \to f(s)$. By (b), $f(s) \leq c$. Similarly (b_i) has a limit $t \geq s$ and $f(t) \geq c$. By (d), $s - t \leq 2^i(b-a) \to 0$, so by the squeeze law $s = t$. Thus $f(t) = f(s) = 0$.

Theorem: If $f\colon [a,b] \to \mathbf{R}$ is continuous and 1-1, then f is strictly monotone.

Proof: Since f is 1-1, it is enough to show monotone. Without loss of generality we may assume that $f(a) < f(b)$ and show that f is increasing. If not, there exist $x < y$ in $[a,b]$ such that $f(x) > f(y)$. If $f(x) > f(b)$, we have a "switch": three points $\{a, x, b\}$ where the extreme value of f occurs at the middle point. Pick c between the extreme value and the closest other value (in our case, pick c between $f(x)$ and $f(b)$) and apply IVT to obtain s_1 and s_2 on opposite sides of the middle point such that $f(s_1) = f(s_2) = c$. Since f is 1-1, this is a contradiction. If $f(x) \leq f(b)$, we again have a switch, this time $\{x, y, b\}$, and a contradiction.