Heat equation in 1-d via the Fourier transform

Heat equation in one spatial dimension: $u_t = c^2 u_{xx}$

Initial condition: $u(x,0) = f(x)$, where $f(x)$ decays at $x = \pm\infty$.

Fourier transform: Transforming the equation we obtain an ODE: $\widehat{u}_t = -c^2 w^2 \widehat{u}$ with gen. solution: $\widehat{u}(w,t) = C(w)\, e^{-c^2 w^2 t}$
At $t = 0$ we obtain $\widehat{u}(w,0) = \widehat{f}(w) = C(w)$, so $\widehat{u}(w,t) = \widehat{f}(w)\, e^{-c^2 w^2 t}$
Therefore $u(x,t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \widehat{u}(w,t)\, e^{iwx}\, dw = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \widehat{f}(w)\, e^{-c^2 w^2 t + iwx}\, dw$ [1]

Separation of variables: We can arrive at the same conclusion using separation of variables.
Let $u(x,t) = F(x)G(t)$. Then $FG_t = c^2 F_{xx} G$, so $\dfrac{1}{c^2}\dfrac{G_t}{G} = \dfrac{F_{xx}}{F} = k$, where k is a constant.
We get a pair of ODEs: $F_{xx} = kF$ and $G_t = c^2 kG$.
By the entropy principle G must decay as t increases, so we must assume that $k < 0$.
Let $k = -w^2$. Then $F_{xx} + w^2 F = 0$ and $G_t + c^2 w^2 G = 0$ [see §11.5 (6) and (7), p. 601 with $p = w$].
Solving both ODEs we obtain $u(x,t;w) = \left[C_1(w)\, e^{iwx} + C_2(w)\, e^{-iwx}\right] e^{-c^2 w^2 t}$,
so the general solution is $u(x,t) = \displaystyle\int_0^{\infty} u(x,t;w)\, dw = \int_0^{\infty} C_1(w)\, e^{-c^2 w^2 t + iwx}\, dw + \int_0^{\infty} C_2(w)\, e^{-c^2 w^2 t - iwx}\, dw$
Changing variable in the second integral $w \mapsto -w$ we obtain

$$u(x,t) = \int_0^{\infty} C_1(w)\, e^{-c^2 w^2 t + iwx}\, dw - \int_0^{-\infty} C_2(-w)\, e^{-c^2 w^2 t + iwx}\, dw = \int_0^{\infty} C_1(w) e^{-c^2 w^2 t + iwx}\, dw + \int_{-\infty}^{0} C_2(-w)\, e^{-c^2 w^2 t + iwx}\, dw$$

Recombining the two integrals we obtain: $u(x,t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \widehat{f}(w)\, e^{-c^2 w^2 t + iwx}\, dw$, where $\widehat{f}(w) = \begin{cases} \sqrt{2\pi}\, C_1(w) & \text{for } w > 0 \\ \sqrt{2\pi}\, C_2(-w) & \text{for } w < 0 \end{cases}$
(You can see that the combined function is $\widehat{f}$ by evaluating at $t = 0$.)

Simplification: We would like to express $u(x,t)$ directly in terms of f rather than $\widehat{f} = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$.

Substitution of the last formula (with dummy variable v) yields $u(x,t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v)\, e^{-iwv}\, dv\right] e^{-c^2 w^2 t + iwx}\, dw$,

Using Fubini's theorem to change the order of integration we obtain $u(x,t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} f(v) \left[\int_{-\infty}^{\infty} e^{-c^2 w^2 t + iw(v-x)}\, dw\right] dv$,

Since $\mathscr{F}\!\left(e^{-ax^2}\right) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-ax^2 - iwx}\, dx = \dfrac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$ [see §10.11 III.9, p. 578], the inner integral evaluates to $\dfrac{\sqrt{\pi}}{c\sqrt{t}} e^{-\frac{(v-x)^2}{4c^2 t}}$,
so $u(x,t) = \dfrac{1}{2c\sqrt{\pi t}} \displaystyle\int_{-\infty}^{\infty} f(v)\, e^{-\frac{(v-x)^2}{4c^2 t}}\, dv$ [see §11.6 (11), p. 611].

Reference: [1] E. Kreyszig, *Advanced Engineering Mathematics*, 8th ed., Wiley, 1999.

[1] For the real Fourier transform version of this see §11.6, pp. 610–611.