Green's identities

(George Green, 1793–1841)

[0,2] with $f = \varphi$ and $p = \nabla\psi$: $\quad \nabla \cdot (\varphi \nabla \psi) = \varphi \nabla^2 \psi + (\nabla\varphi) \cdot (\nabla\psi)$

$$\text{I.} \quad \int_\Omega (\varphi \nabla^2 \psi + \nabla\varphi \cdot \nabla\psi)\, dV = \int_{\partial\Omega} \varphi (\nabla\psi) \cdot\, ds$$

$$\text{II.} \quad \int_\Omega (\varphi \nabla^2 \psi - \psi \nabla^2 \varphi)\, dV = \int_{\partial\Omega} (\varphi \nabla\psi - \psi \nabla\varphi) \cdot\, ds$$

$$\text{III.} \quad \varphi = -\frac{1}{4\pi} \int_\Omega \frac{\nabla^2 \varphi}{|\rho|}\, dV + \frac{1}{4\pi} \int_{\partial\Omega} \left[\frac{1}{|\rho|} \nabla\varphi - \varphi \nabla \frac{1}{|\rho|} \right] \cdot ds$$

Uniqueness Theorem: A harmonic C^1 function is uniquely determined up to an additive constant by the values of its normal derivative at the boundary.

Proof: Green's first identity with harmonic $\varphi = \psi$ gives

$$\int_\Omega (\nabla\varphi)^2\, dV = \int_{\partial\Omega} \varphi (\nabla\varphi) \cdot\, ds$$

Thus, if the normal derivative of φ vanishes at the boundary, $\varphi = \text{const.}$

Representation Theorem: Any harmonic C^1 function is representable as a superposition of potentials due to distributions of sources and doublets.

Proof: Green's third identity.