Fundamental theorem of calculus:
Given a k dimensional chain Ω and a degree $k-1$ differential form ω, we have $\displaystyle\int_{\partial\Omega}\omega=\int_\Omega d\omega$.

Proof of FTC for the k dimensional unit box:
Since the integral and the exterior derivative are linear, we may assume $\omega = a(x)\,\widehat{dx_i}$. Then $d\omega=(-1)^{i+1}\partial_i a\,dV$.
Only the 2 faces orthogonal to the i-th axis contribute to

$$\int_{\partial\Omega}\omega=(-1)^i\left[\int_{F_{i0}}\omega-\int_{F_{i1}}\omega\right].$$

On the other hand, by Fubini's theorem and the univariate FTC,

$$\int_\Omega d\omega=(-1)^{i+1}\int_\Omega\partial_i a\,dV=(-1)^{i+1}\int_\Omega\partial_i a\,dV=(-1)^{i+1}\left[\int_{F_{i1}}\omega-\int_{F_{i0}}\omega\right].$$

Pullbacks: Given $\varphi\colon U\to\Omega$, define $\varphi^*f=f\circ\varphi$, $\varphi^*dx_i=d\varphi_i$. and extend to all forms by

- $\varphi^*(\alpha+\omega)=\varphi^*\alpha+\varphi^*\omega$
- $\varphi^*(f\omega)=(f\circ\varphi)\,\varphi^*\omega$
- $\varphi^*(\alpha\wedge\omega)=\varphi^*\alpha\wedge\varphi^*\omega$

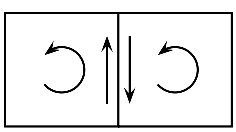
Applying $\varphi^*\omega$ at a point u in U to k vectors is the same as applying ω to the images of the vectors:
$(\varphi^*\omega)_u[v_1,...v_k]=\omega_{\varphi(u)}[d\varphi_u(v_1),...d\varphi_u(v_k)]$.
Furthermore,

- $\displaystyle\int_\Omega\omega=\int_U\varphi^*\omega$
- Chain rule: $d(\varphi^*\omega)=\varphi^*(d\omega)$ [Proof by induction on k.]

Proof of FTC for a k cell: Parametrize the cell Ω by $\varphi\colon I^k\to\Omega$. Then

$$\int_\Omega d\omega=\int_{I^k}\varphi^*(d\omega)=\int_{I^k}d(\varphi^*\omega)=\int_{\partial I^k}\varphi^*\omega=\sum_{i=1}^k(-1)^i\left[\int_{F_{i0}}\varphi^*\omega-\int_{F_{i1}}\varphi^*\omega\right]=\sum_{i=1}^k(-1)^i\left[\int_{\varphi(F_{i0})}\omega-\int_{\varphi(F_{i1})}\omega\right]=\int_{\partial\Omega}\omega.$$

Proof for an oriented cellular region:



Reference:
C. H. Edwards, Jr., *Advanced calculus of several variables*, Academic Press, 1973 (Dover, 1994)