Permutations, determinant, cross product

Permutations: Consider the set with n elements $I_n = \{1, ...n\}$. A permutation is a 1-1 onto map $s\colon I_n \to I_n$. The set of all permutations is denoted by S_n and has $n!$ permutations (there are n choices for $s(1)$, $n-1$ choices for $s(2)$ etc.).
Each permutation can be written as a composition of transpositions. This representation is not unique, but the parity is unique and is denoted by $\operatorname{sgn}(s)$.

Determinant: Given an $n \times n$ matrix $A = (a_{ij})$, the determinant of A is $\det(A) = \sum_{s \in S_n} \operatorname{sgn}(s) \prod_{i=1}^{n} a_{i\,s(i)}$.

n	$\det(A)$
1	a_{11}
2	$a_{11}a_{22} - a_{12}a_{21}$
3	$a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$

Properties of Determinant:

1. $\det(A) = \det(A^T)$, where $A^T = (a_{ji})$ is the *transpose* of A.

   This means that in all statements about the determinant rows may be replaced by columns.

2. Determinant is an *alternating multilinear* function of the rows of A.

   Alternating means that a permutation of the rows introduces a sign change according to its parity.
   Multilinear means that it is linear in each variable.
   For example $\det\begin{pmatrix}1 & 2\\ 5 & 4\end{pmatrix} = -\det\begin{pmatrix}5 & 4\\ 1 & 2\end{pmatrix} = \det\begin{pmatrix}1 & 2\\ 3 & 0\end{pmatrix} + 2\det\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix} = \det\begin{pmatrix}1 & 2\\ 3 & 0\end{pmatrix} = -6$

3. $\det(AB) = \det(A)\det(B)$.

Laplace expansion: If we collect terms by fixing the first index we obtain what is known as a Laplace [1] or *cofactor* expansion along the corresponding row. Each element of the row is multiplied by the determinant of the *cofactor* — the matrix obtained by crossing out the row and the column. Start writing the cofactor downwards and to the right from the entry and wrap around as needed. The results are added.
For example for $n = 3$ we choose the second row: $\det(A) = a_{21}\det\begin{pmatrix}a_{32} & a_{33}\\ a_{12} & a_{13}\end{pmatrix} + a_{22}\det\begin{pmatrix}a_{33} & a_{31}\\ a_{13} & a_{11}\end{pmatrix} + a_{23}\det\begin{pmatrix}a_{31} & a_{32}\\ a_{11} & a_{12}\end{pmatrix}$.
Geometric Interpretation: Determinant gives n-dimensional volume of the parallelepiped formed by the rows of A considered as vectors in $\mathbf{R}^n$. It may be negative, so we call it "signed" volume.

n	$\det(A)$
1	"signed" length of a_{11}
2	"signed" area of the parallelogram with sides (a_{11}, a_{12}) and (a_{21}, a_{22})
3	"signed" volume of the parallelepiped with sides (a_{11}, a_{12}, a_{13}),(a_{21}, a_{22}, a_{23}),(a_{31}, a_{32}, a_{33}).

Cross product: In $\mathbf{R}^3$ we define $u \times v$ as the vector of cofactors for the first row expansion of $\det\begin{pmatrix}? & ? & ?\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{pmatrix}$.

$$u \times v = \left(\det\begin{pmatrix}u_y & u_z\\ v_y & v_z\end{pmatrix}, \det\begin{pmatrix}u_z & u_x\\ v_z & v_x\end{pmatrix}, \det\begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix}\right) = \det\begin{pmatrix}\widehat{\imath} & \widehat{\jmath} & \widehat{k}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{pmatrix}.$$

$|u \times v|$ is the "signed" area of the parallelogram given by u and v. $u \times v$ is $\perp$ to the plane of u and v.
We can complete the expansion into a full determinant by placing another vector w in place of the question marks.
This is sometimes known as the triple product $(wuv) = w \cdot (u \times v)$.

References

[1] M. Marcus, H. Minc, *A survey of matrix theory and matrix inequalities*, Dover, 1992 [Section 2]

[2] B. Jacob, *Linear Algebra*, Freeman, 1990 [Chapter 2]

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