# Spaces of complex valued functions

If $G$ is a nonempty set, let $\mathbf{C}^G := \{u\colon G \to \mathbf{C}\}$.
With pointwise algebraic operations $\mathbf{C}^G$ inherits ring structure from $\mathbf{C}$. In particular, $\mathbf{C}^G$ is a complex vector space.
For $G := \{k \in \mathbf{Z}\colon 1 \le k \le n\}$, $\mathbf{C}^G = \mathbf{C}^n$ with $u_k := u(k)$.

**Inner product (dot product) on $\mathbf{C}^n$:** $\langle u, v\rangle := \sum_{k=1}^{n} u_k \overline{v}_k$

- positive: $\langle u, u\rangle \ge 0$, definite: $\langle u, u\rangle = 0 \Rightarrow u = 0$
- conjugate symmetric: $\langle u, v\rangle = \overline{\langle v, u\rangle}$
- conjugate bilinear: $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$, $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$, $\langle cu, v\rangle = c\langle u, v\rangle$, $\langle u, cv\rangle = \overline{c}\langle u, v\rangle$

**Norms on $\mathbf{C}^n$:** $p$-norm: $|u|_p := \left(\sum_{k=1}^{n} |u_k|^p\right)^{1/p}$, where $p \ge 1$; $\quad |u|_\infty = \max\{|u_k|\colon 1 \le k \le n\}$

- positive: $|u|_p \ge 0$, definite: $|u|_p = 0 \Rightarrow u = 0$
- Hölder [1] inequality: $|\langle u, v\rangle| \le |u|_p\,|v|_q$, where $p + q = pq$

  Proof ($p = 2$): $0 \le \langle u - cv, u - cv\rangle = \langle u, u\rangle - \overline{c}\langle u, v\rangle - c\langle v, u\rangle + c\overline{c}\langle v, v\rangle$, let $c = \langle u, v\rangle / \langle v, v\rangle$.
- Minkowski inequality (triangle inequality): $|u+v|_p \le |u|_p + |v|_p$

  Proof: $\sum_{k=1}^{n} |u_k + v_k|^p = \sum_{k=1}^{n} |u_k|\,|u_k+v_k|^{p-1} + \sum_{k=1}^{n} |v_k|\,|u_k+v_k|^{p-1}$, apply Hölder.
- Polarization identity: $|u+v|_2^2 + |u-v|_2^2 = 2\left(|u|_2^2 + |v|_2^2\right)$, so $4\langle u, v\rangle = |u+v|_2^2 - |u-v|_2^2$

**Complex valued functions on topological groups:** In the definition of inner product we "sum" over $G$, so $G$ must have a measure. We will look at topological groups with Haar measure. Specifically we are interested in the following examples:

- $G = \mathbf{Z}_+$ or $G = \mathbf{Z}$ ($\mathbf{C}^G$ = infinite sequences)
- $G = \mathbf{T}$ := the unit circle $S^1 \subset \mathbf{C}$ with $d\sigma = d\theta/(2\pi)$ ($\mathbf{C}^G$ = periodic functions of a real variable)
- $G = \mathbf{R}$ with $dx$ ($\mathbf{C}^G$ = functions of a real variable)

**Lebesgue spaces:** For discrete $G$, $\ell_p := \left\{u \in \mathbf{C}^G\colon \sum_{k\in G} |u_k|^p < \infty\right\}$
Definitions of $p$-norm are the same as above. In $\ell_2$, in view of Hölder's inequality, we may define an inner product as well.

For continuous $G$, $L_p(G) := \left\{u \in \mathbf{C}^G\colon \int_G |u(t)|^p\,dt < \infty\right\} / \sim$, where $u \sim v \Leftrightarrow \{t \in G\colon u(t) \ne v(t)\}$ has measure zero.

Norm on $L_p(G)$: $|u|_p := \left(\int_G |u(t)|^p\,dt\right)^{1/p}$. Inner product on $L_2(G)$: $\langle u, v\rangle := \int_G u(t)\overline{v(t)}\,dt$

*Theorem:* $L_2(\mathbf{T}) \subset L_1(\mathbf{T})$

*Proof:* $|f|_1 = \left\langle f, \overline{f}/|f|\right\rangle$

**Riesz-Fischer Theorem (Cauchy criterion for $L_2$):** $L_2$ is complete: a sequence $u_n \in L_2$ converges in the mean, i.e. $\exists u \in L_2$ with $|u - u_n|_2 \to 0, \Leftrightarrow \forall\varepsilon > 0\ \exists N$ such that $n, m \ge N \Rightarrow |u_n - u_m|_2 < \varepsilon$.

**Weak convergence:** In $L_2$, $u_n \to u$ weakly means $\forall v \in L_2\ \langle u_n, v\rangle \to \langle u, v\rangle$.

*Theorem:* $u_n \to u$ weakly $\Rightarrow |u|_2 \le \liminf |u_n|_2$.

*Proof:* Suppose $|u_n|_2 \le b$ for large $n$. Then $|u|_2^2 = \langle u, u\rangle = \lim\langle u_n, u\rangle \le b\,|u|_2$, so $|u|_2 \le b$.

*Theorem:* $u_n \to u$ in the mean $\Leftrightarrow u_n \to u$ weakly and $|u_n|_2 \to |u|_2$.

*Proof:* $|\langle u_n, v\rangle - \langle u, v\rangle| = |\langle u_n - u, v\rangle| \le |u_n - u|_2\,|v|_2$, $\left||u_n|_2 - |u|_2\right| \le |u_n - u|_2$.
Conversely $|u - u_n|_2^2 = |u_n|_2^2 - \langle u_n, u\rangle - \langle u, u_n\rangle + |u|_2^2 \to |u|_2^2 - \langle u, u\rangle - \langle u, u\rangle + |u|_2^2 = 0$.

**Theorem of Choice:** $L_2$ is separable and every bounded sequence has a weakly convergent subsequence.

**Reference:** F. Riesz, B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar, 1955 (Dover, 1990).



[1] The extension of this and Minkowski inequality to integrals is due to F. Riesz. Special case for $p = 2$ is known as Cauchy inequality and its extension to integrals, known as Schwartz inequality, is due to Bunyakovsky.