Functions $\mathbf{C} \to \mathbf{C}$ as plane transformations

Complex numbers: Complex numbers [1] are written in the form $z = a + ib$, where a and b are real numbers and i is a symbol [2] satisfying $i^2 = -1$. We call a the real part of z and b the imaginary part of z written: $a = \operatorname{Re} z$ and $b = \operatorname{Im} z$. By identifying $a + ib$ with $\begin{bmatrix} a \\ b \end{bmatrix}$ we can interpret $\mathbf{C} \stackrel{\text{def}}{=} \{a + ib\colon a, b \in \mathbf{R}\}$ as $\mathbf{R}^2$ — the real plane [Caspar Wessel, 1799]. In polar coordinates $z = r(\cos\theta + i\sin\theta)$, where r is the magnitude and θ is the angle, written: $r = |z|$, $\theta = \arg z$.

Complex arithmetic: We can generalize real addition and multiplication [3] to $\mathbf{C} \stackrel{\text{def}}{=} \{a + ib\colon a, b \in \mathbf{R}\}$ such that they satisfy their usual properties. If $z = a + ib$ and $w = c + id$, then $z + w = (a + c) + i(b + d)$ and $zw = (ac - bd) + i(bc + ad)$. We can generalize real absolute value $|x| = \sqrt{x^2}$ to modulus $|z| = \sqrt{a^2 + b^2}$. This is just the magnitude of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$. As such, it is positive definite ($z = 0 \Leftrightarrow |z| = 0$) and satisfies the triangle inequality $|z + w| \le |z| + |w|$. Note that $|z|^2 = z\overline{z}$, where $\overline{z} = a - ib$ is the complex conjugate of z. Thus, if $z \neq 0$, it is a unit and $1/z = \overline{z}/|z|^2$. I.e. $\mathbf{C}$ is a field extension of $\mathbf{R}$.

Complex multiplication as a linear transformation: Given $z = a + ib \in \mathbf{C}$, the plane transformation $w \mapsto zw$ is linear. In Cartesian coordinates it is represented by the orthogonal matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

Thus, we may think of complex numbers as matrices in above form and complex operations can now be interpreted through familiar matrix operations. Complex addition and multiplication are just matrix addition and multiplication. Complex conjugation corresponds to transpose and the square of modulus is determinant. It follows immediately that $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$, and $|zw| = |z|\,|w|$. The fact that $1/z = \overline{z}/|z|^2$ is a manifestation of Cramer's rule.

Converting to polar coordinates we obtain $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. This shows that complex multiplication is an isotropic dilation by r composed (in either order) with a rotation by θ.

Geometry of complex operations:

- $z \to \overline{z}$ is the reflection about the real axis
- 0, z, w, and $z + w$ are vertices of a parallelogram
- $|zw| = |z|\,|w|$ and $\arg(zw) = \arg(z) + \arg(w)$
- $|z^n| = |z|^n$ and $\arg(z^n) = n\arg(z)$
- $z \to 1/z$ is the unit circle inversion followed by conjugation

Differentiable functions: Suppose $f\colon \mathbf{C} \to \mathbf{C}$ and $z \in \mathbf{C}$. Define $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z + \Delta z) - f(z)}{\Delta z}$. If $f'(z)$ exists, we say f is differentiable at z. This is equivalent to the approximation statement $f(z + \Delta z) = f(z) + f'(z)\,\Delta z + \varepsilon$, where $\varepsilon/\Delta z \to 0$ as $\Delta z \to 0$. The linear function of $\Delta z \to f'(z)\,\Delta z$ is called the differential of f (written df). [4] The usual rules of differentiation can be proved, however the mean value theorem and l'Hôpital's rule do not in general hold.
A function $f\colon \mathbf{C} \to \mathbf{C}$ can also be viewed as a plane transformation by writing $z = x + iy$ and $f(z) = u + iv$, so $z \to f(z)$ becomes $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$. The differential becomes $\begin{bmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$. The derivative matrix represents a complex number $f'(z)$, if it has the required form, i.e. it satisfies the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$. Thus f is differentiable at $z \Leftrightarrow$ it is differentiable as a real plane transformation at z and satisfies the Cauchy-Riemann equations. [5] A function differentiable in a neighbourhood of z is called holomorphic at z. [6]

Exponentials and logs: Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$ (think of this as a definition, but note that Maclaurin series look right). Then, $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$. Trig is dead! Now define $e^z = e^{x+iy} = e^x e^{iy}$, so $\log z = \ln|z| + i\arg z$ (beware: arg is multivalued).

[1] First appeared in *Ars Magna* (1545) by Girolamo Cardano (1501–1576). *Ars Magna* incorporated the work on the solution of cubic and quartic equations by Tartaglia (Niccolo Fontana 1499–1557), the author, and his assistant Lodovico Ferrari (1522–1565). Cardano's opinion was that complex numbers "are as subtle as they are useless".

[2] $\sqrt{-1}$ is denoted i by mathematicians and j by physicists and engineers. Square roots of negative real numbers have no meaning in the real domain, yet were useful in formally manipulating formulas for the solutions of polynomial equations.

[3] Complex arithmetic was worked out in *l'Agebra* (1560, pub. 1572) by Rafael Bombelli (1526–1572), who used complex numbers to find *real* roots of certain quartics.

[4] In the special case when $f(z) = z$ we have $f' = 1$, so dz is the identity map $\Delta z \to \Delta z$. In other words, $dz(\Delta z) = \Delta z$. Thus, in general, $df(\Delta z) = f'(z)\,\Delta z = f'(z)\,dz(\Delta z)$. Dropping the explicit dependence on Δz we write $df = f'(z)\,dz$.

[5] A plane transformation whose partial derivatives exist and are continuous in a neighbourhood of z is differentiable at z. If it satisfies the Cauchy-Riemann equations then it is complex differentiable at z. In this case, the requirement that the partials must be *a priori* continuous may be dropped [Looman-Menschoff theorem].

[6] This turns out to be equivalent to analyticity (convergence of Taylor series) in a neighborhood of z, so the terms holomorphic and analytic are used interchangeably.