Complex Calculus

Complex numbers: Let $R = \mathbf{R}[X]$ be the univariate polynomial ring $\mathbf{R}[X]$ (free commutative $\mathbf{R}$-algebra on $\{X\}$). It is a principal ideal domain (III.6.4 [2]), so the ideal M generated by X^2+1 is maximal and $\mathbf{C} \overset{\text{def}}{=} R/M$ is a field. We let $i = [X]$. If $z \in \mathbf{C}$, then uniquely $z = a + ib$, where $a, b \in \mathbf{R}$, so $\mathbf{C} \cong \mathbf{R}^2$ as a complete normed real vector space, where we define the *complex conjugate* $\overline{a+ib} = a - ib$. and *modulus* $|z| \overset{\text{def}}{=} \sqrt{z\bar{z}}$. In polar coordinates $z = r\cos\theta + ir\sin\theta$, where $r = |z|$ and $\theta = \arg z$ (angle). Multiplication adds angles, suggesting exponential notation $z = re^{i\theta}$ (confirmed by Taylor series).

Differentiation: The differential df of a complex function $f(z)$ is a $\mathbf{C}$-linear map of Δz that approximates $\Delta f = f(z+\Delta z) - f(z)$ at z. We write $\Delta f = df + \varepsilon$ and require that with z fixed, $\varepsilon/\Delta z \to 0$ as $\Delta z \to 0$. A $\mathbf{C}$-linear map of Δz must be of the form $\Delta z \mapsto c\Delta z$, so $df = c(z)\Delta z$. The coefficient $c(z)$ is called the derivative of f and is denoted f'. Since $\varepsilon/\Delta z \to 0$, $f'(z) = \lim\limits_{\Delta z \to 0} \Delta f/\Delta z$ (see Theorems 3.1–2 [4]). If df exists on a *domain*, i.e. an open connected (thus, path connected) set $\Omega \subseteq \mathbf{C}$, f is called *holomorphic* ($f \in \mathcal{H}(\Omega)$). In fact, $f \in \mathcal{H}(\Omega) \Leftrightarrow f$ is *analytic* (locally representable by power series).

Cauchy-Riemann equations: We can consider f as a real vector function by letting $z = x + iy$ and $f = u + iv$, where u and v are real functions of x and y. Then $df = du + i\,dv = (u_x\,dx + u_y\,dy) + i(v_x\,dx + v_y\,dy)$. This is a $\mathbf{C}$-linear map of $dz = dx + i\,dy \Leftrightarrow a \overset{\text{def}}{=} u_x = v_y$ and $b \overset{\text{def}}{=} v_x = -u_y$. In this case $df = (a+ib)(dx + i\,dy)$, so $f' = a + ib$.

Looman-Menschoff theorem: If f is holomorphic, we get the Cauchy-Riemann equations $u_x = v_y$, $v_x = -u_y$. Conversely if u_x, u_y, v_x, v_y exist and are continuous, then f is differentiable as a real vector function. Let $df = du + i\,dv$. The C-R equations $\Rightarrow df$ is $\mathbf{C}$-linear, so f is holomorphic. In fact, we need not require the continuity of the partials (see 1.6 [3]).

Properties of differentiation: For algebraic operations and composition the rules are the same as in real calculus.

Curves and partitions: Let Ω be a domain and let $[a,b] \subseteq \mathbf{R}$ and $c\colon [a,b] \to \Omega$ be continuous. The image of c is a curve in Ω. A partition of $[a,b]$ is a finite subset containing the endpoints. For a partition $P = \{a_0 = a < a_1 < \ldots < a_n = b\}$ of $[a,b]$, define $|P| = \max\limits_{k=0}^{n-1}(a_{k+1} - a_k)$. The set of all partitions is a directed set and $P \subseteq Q \Rightarrow |P| \geq |Q|$.

Riemann-Stieltjes sums: Given a partition P, choose $a_k^* \in [a_k, a_{k+1}]$. Let $z_k = c(a_k)$, $\Delta z_k = z_{k+1} - z_k$, $z_k^* = c(a_k^*)$, $L_P = \sum\limits_{k=0}^{n-1} |\Delta z_k|$, and $S_P = \sum\limits_{k=0}^{n-1} f(z_k^*)\Delta z_i$. If c has bounded variation, i.e. $|L| \overset{\text{def}}{=} \int_L |dz| \overset{\text{def}}{=} \sup\limits_P L_P < \infty$, L is called *rectifiable*.

Integrals: Since $[a,b]$ is compact and c is continuous, L is compact. If f is continuous ($f \in \mathcal{C}(\Omega)$), then it is uniformly continuous on L, so if $\varepsilon > 0$, $\exists\delta > 0$ with $|w_1 - w_2| < \delta \Rightarrow |f(w_1) - f(w_2)| < \varepsilon$. Let $\varepsilon_m \to 0$ monotonically and let I_m be the closure of $\{S_P\colon\ |P| < \delta_m\}$. Then $I_m \supseteq I_{m+1}$ and $\operatorname{diam} I_m \leq 2\varepsilon_m\,|L| \to 0$, so by Cantor's Theorem $\bigcap\limits_{m=1}^{\infty} I_m = \{I\} \overset{\text{def}}{=} \int_L f(z)\,dz$ (see IV.1.4 [1]). The integral I does not depend on the choice of parametrization c (see IV.1.13 [1]).

Example: If $c(t)$ is smooth, $dz = c'dt$. Let $c(t) = e^{it}$, $-\pi < t \leq \pi$ (unit circle) and $f(z) = 1/z$. Then $dz = ie^{it}\,dt$ and $\int_c f(z)\,dz = i\int_{-\pi}^{\pi} f(e^{it})\,e^{it}\,dt = i\int_{-\pi}^{\pi} \frac{1}{e^{it}}\,e^{it}\,dt = i\int_{-\pi}^{\pi} dt = 2\pi i.$

Properties of integration: The integral is linear in f and additive in L. If $|f| \leq M$ on L, then $\left|\int_L f(z)\,dz\right| \leq M\,|L|$.

Cauchy-Goursat-Morera theorem: If $f \in \mathcal{C}(\Omega)$, then $f \in \mathcal{H}(\Omega) \Leftrightarrow \int_L f(z)\,dz = 0$ for all boundary L ($[L] = 0 \in H_1(\Omega)$).
Proof: If $f \in \mathcal{H}(\Omega)$, then $f(z)\,dz$ is closed. Indeed, $df = f'\,dz$, so $d(f\,dz) = f'\,dz \bigwedge dz = 0$. Since $[L] = 0$, there exists a 2-chain $D \subseteq \Omega$ with $\partial D = L$. If f' is continuous, Green's theorem shows $\int_L f(z)\,dz = \int_D d(f(z)\,dz) = 0$. Continuity of f' need not be assumed (Goursat) (see e.g. Theorem 1.2.2 [3]). To prove the converse (Morera) we may assume that Ω is a disc. Let $w_0, w \in \Omega$. If L_1, L_2 are paths from w_0 to w, then $L_2 - L_1$ is a boundary. Thus, $F(w) \overset{\text{def}}{=} \int_{w_0}^{w} f(z)\,dz$ is path independent and $F' = f$. But Cauchy's integral formula (below) implies that F' is differentiable.

Deformation principle: If $f \in \mathcal{H}(\Omega)$ and $[L_1] = [L_2]$, then $I(L_1, f) = I(L_2, f)$. We get a bilinear map $I\colon H_1(\Omega) \times \mathcal{H}(\Omega) \to \mathbf{C}$.

Cauchy's Integral Formula: If $g \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$ and L is a boundary simple closed rectifiable oriented curve around z_0, then $\int_L \frac{g(z)}{(z-z_0)^{k+1}}\,dz = 2\pi i\,c_k$, where $c_k = \frac{g^{(k)}(z_0)}{k!}$. In particular, g is $\mathcal{C}^\infty$ and c_k are its Taylor coefficients.
Proof: Deform L to a circle of radius r around z_0. Let $\varepsilon > 0$. By continuity of g, $\exists r > 0$ with $|z - z_0| \Rightarrow |g(z) - g(z_0)| < \varepsilon$. Then $\left|\int_L \frac{g(z) - g(z_0)}{z - z_0}\,dz\right| \leq 2\pi\varepsilon$. Since ε is arbitrary, $\int_L \frac{g(z)}{z - z_0}\,dz = \int_L \frac{g(z_0)}{z - z_0}\,dz = g(z_0)\int_L \frac{1}{z - z_0}\,dz = g(z_0)\,2\pi i.$
To obtain the general formula, differentiate both sides with respect to z_0.

References:

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[2] T. Hungerford, *Algebra*, Holt, Rinehart and Winston, 1974

[3] R. Narasimhan, *Complex analysis in one variable*, Birkhäuser, 1985

[4] R. Silverman, *Introductory complex analysis*, Dover, 1972