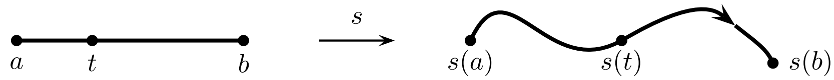
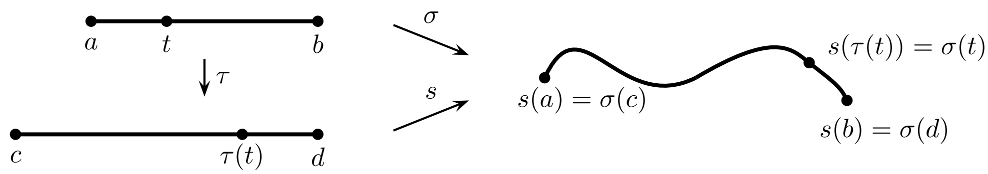
Parametric curves and integration

Parametrization: Suppose $s\colon [a,b] \to \mathbf{R}^n$ is a smooth parametric curve. Intuitively it helps to think of the parameter t as time ($a \le t \le b$) and $s(t)$ as a position vector (a point) in space at a given time t. As t varies from a to b, $s(t)$ traces out a geometric curve in $\mathbf{R}^n$ from one endpoint to the other: from $s(a)$ to $s(b)$. The direction of travel is called *orientation* and is usually expressed by drawing an arrow along the curve.



Reparametrization: A different parametrization of the same geometric curve can be obtained by smoothly slowing down or speeding up t. We introduce a new time parameter τ which depends smoothly and monotonically on t (and vice versa). In other words, $\tau\colon [a,b] \to [c,d]$ is an invertible function of t in the category of smooth maps. Such new time τ is called a *reparametrization* (cf. Def. 6.1.3, p. 378). By abuse of notation let us use τ to denote both the new parameter and the function. The two parameters are related by $\tau = \tau(t)$ and $t = \tau^{-1}(\tau)$. We get a new parametrization of the curve $\sigma\colon [a,b] \to \mathbf{R}^n$ by substituting the new time τ into the original formula: $\sigma(t) = s(\tau(t))$, i.e. $\sigma = s \circ \tau$ (see Example 6, p. 378).



Orientation: If a reparametrization τ is increasing with t, it is called *orientation preserving* and if τ is decreasing with t — *orientation reversing* (cf. p. 379).

Relating different parametrizations: Given two different parametrizations s and σ of the same curve, finding the corresponding reparametrization τ can sometimes be done by inspection, as in Example 6, p. 378. The equation $\sigma = s \circ \tau$ (see above) is an implicit formula for τ. Explicitly $\tau(t) = s^{-1}(\sigma(t))$, i.e. $\tau = s^{-1} \circ \sigma$ (see the diagram above).

Integration: We can integrate vector or scalar fields along a curve by reducing the problem to integration with respect to a single parameter t:

$$\int F \cdot ds = \int F(s(t)) \cdot d(s(t)) = \int_a^b F(s(t)) \cdot s'(t)\, dt, \qquad \int f\,|ds| = \int f(s(t))\,|d(s(t))| = \int_a^b f(s(t))\,|s'(t)|\; dt.$$

Invariance of integration: Integration along a curve does not depend on parametrization (except possibly for sign). Suppose τ is orientation preserving. Then $\tau(a) = c$ and $\tau(b) = d$, so using the substitution $\tau = \tau(t)$

$$\int F \cdot d\sigma = \int_a^b F(\sigma(t)) \cdot d(\sigma(t)) = \int_a^b F(s(\tau(t))) \cdot d(s(\tau(t))) = \int_c^d F(s(\tau)) \cdot d(s(\tau)) = \int F \cdot ds.$$

Suppose τ is orientation reversing, then $\tau(a) = d$ and $\tau(b) = c$, so we need a minus sign to straighten the situation out. The case of scalar field integration is handled similarly.

Application: One reparametrization, often used in computational mathematics, is reparametrization by arclength. We let $\tau(t)$ be the arclength between $s(a)$ and $s(t)$:

$$\tau(t) = \int_{s(a)}^{s(t)} |ds| = \int_a^t |s'(t)|\; dt$$

By the Fundamental Theorem of Calculus $\tau'(t) = |s'(t)|$. Therefore, with the new time τ the speed is

$$\left|\frac{ds}{d\tau}\right| = \left|\frac{ds}{dt}\frac{dt}{d\tau}\right| = \left|\frac{ds}{dt}\right| \Big/ \left|\frac{d\tau}{dt}\right| = |s'(t)| \,/\, |s'(t)| = 1.$$

One useful consequence is the fact that acceleration $d^2s/d\tau^2$ is perpendicular to the curve (i.e. perpendicular to $ds/d\tau$, which is tangent to the curve). This follows immediately by implicit differentiation of the equation $(ds/d\tau) \cdot (ds/d\tau) = 1$.

Interpretation: Integrals along curves often occur in physics. For example, if f is linear density along the curve, then $f\,|ds| = dm$, where m is mass. To interpret vector field integration let us parametrize by arclength. Then velocity $ds/d\tau$ is a unit vector, so $F \cdot ds/d\tau$ is the component of F along the curve. Therefore, the integral of a vector field can be interpreted as scalar integration of the component of the vector along the curve. For example, if F is a force field, then $F \cdot ds = dW$, where W is the work performed by the force along the curve.

Reference: S. J. Colley, *Vector Calculus*, Prentice-Hall, 1999.