Pointwise convergence of Fourier series

Dirichlet kernel: $D_n := \sum_{k=-n}^{n} \exp(ik\theta) = \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{\sin\left[\frac{1}{2}\theta\right]}$, $s_n = D_n * u$. Not a summability kernel!

Fejér's theorem: $\sigma_n \to \check{u}$, where $\check{u}(\theta) = \frac{1}{2}\lim_{h\to 0}\left[u(\theta-h)+u(\theta+h)\right]$.

Proof: K_n is even and $\sup\{K_n(\theta):\ -\delta<\theta<\delta\} \to 0$

$$\sigma_n - \check{u} = K_n * u - \check{u} = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\tau)\left[u(\theta-\tau)-\check{u}(\theta)\right]d\tau = \frac{1}{\pi}\left[\int_0^{\delta}+\int_{\delta}^{\pi}\right]K_n(\tau)\left[\frac{u(\theta-\tau)+u(\theta+\tau)}{2}-\check{u}(\theta)\right]d\tau$$

For the first integral the expression in the brackets is small and for the second K_n is small.

- At points of continuity of u, $\sigma_n \to u$.
- On compact subsets of continuity of u the convergence is uniform.

  Proof: Use uniform continuity of u to choose one δ for all θ.
- Existence of limit in Fejér's theorem can be weakened (Lebesgue) to $\exists\check{u}$ with $\lim_{h\to 0}\int_0^h\left|\frac{u(\theta-\tau)+u(\theta+\tau)}{2}-\check{u}\right|d\tau = 0$.
- This $\check{u} = u$ a.e., so $\sigma_n \to u$ a.e.
- $u \in L_2(\mathbf{T}) \Rightarrow$ Fourier series $\to u$ a.e. (Carleson, 1965)
- $u \in L_p(\mathbf{T}), p > 1 \Rightarrow$ Fourier series $\to u$ in the mean (we have seen this for L_2).
- $\exists$ continuous u whose Fourier series diverges at a point!

Mercer's theorem: [1] $u \in L_1 \Rightarrow \widehat{u}_k \to 0$

Proof: $\forall\varepsilon > 0, \exists$ trig polynomial p with $|u-p|_1 < \varepsilon$. For $n > \deg p$, $|\widehat{u}_n| = \left|(\widehat{u-p})_n\right| \le |u-p|_1$.

Hardy's Tauberian theorem: $u \in L_1(\mathbf{T}), \widehat{u}_k = \mathcal{O}\left(\frac{1}{k}\right) \Rightarrow$ the convergence of s_n is the same as that of σ_n.

Proof: Let $\varepsilon > 0$. $\exists\lambda > 1$ such that $\limsup \sum_{n<|k|\le\lambda n} |\widehat{u}_k| < \varepsilon$.

$$s_n = \frac{[[\lambda n]]+1}{[[\lambda n]]-n}\sigma_{[[\lambda n]]} - \frac{n+1}{[[\lambda n]]-n}\sigma_n - \frac{[[\lambda n]]+1}{[[\lambda n]]-n}\sum_{n<|k|\le\lambda n}\left[1-\frac{|k|}{[[\lambda n]]+1}\right]\widehat{u}_k\exp(ik\theta)$$

For large enough n (dependent only on the rate of convergence of σ_k) the sum $< \varepsilon$, so $|s_n - \lim\sigma_k| < 2\varepsilon$.

- If u is of bounded variation, then the hypothesis (and conclusion) of the Tauberian theorem hold.

  Proof: $|\widehat{u}_k| \le \frac{\mathrm{var}\,(u)}{2\pi\,|k|}$

Reference: Y. Katznelson, *An Introduction to Harmonic Analysis*, Wiley, 1968 (Dover, 1976)

[1] Analogous result for the continuous Fourier transform is known as the Riemann-Lebesgue Lemma.